Graphs of holomorphic functions

$V \subset \mathbb{C}$ open $\quad f : V \to \mathbb{C}$ holomorphic

$$X = \text{graph}(f) = \{ (z, f(z)) \mid z \in V \} \subset \mathbb{C}^2$$

$$\pi : \mathbb{C}^2 \to \mathbb{C} \qquad \pi(z,w) = z$$

Atlas w/ one chart $\quad \varphi = \pi|_X : X \to V$

Smooth affine plane curves

$\mathbb{C}^2 =$ "affine plane"

let $f(z,w) \in \mathbb{C}[z,w]$ be a polynomial in two variables

let $X = V(f) := \{ (z,w) \in \mathbb{C}^2 \mid f(z,w) = 0 \}$
be the <u>vanishing locus of f</u>.

X is called an <u>affine plane curve</u> ("curve" b/c 1-dim over $\mathbb{C}$)

<u>Def</u> f is <u>nonsingular</u> if f, $\frac{\partial f}{\partial z}$, and $\frac{\partial f}{\partial w}$ never vanish simultaneously : i.e.

$$f(p) = 0 \Rightarrow \frac{\partial f}{\partial z}(p) \neq 0 \text{ or } \frac{\partial f}{\partial w}(p) \neq 0$$

If f is nonsingular we call $X = V(f)$ <u>nonsingular</u> or <u>smooth</u>.

<u>Theorem</u> If f is nonsingular then $X = V(f)$ is a Riemann surface. (possibly disconnected)

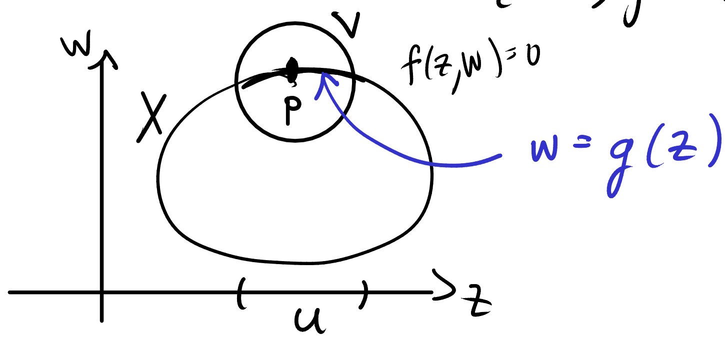
<u>Theorem</u> (Holomorphic implicit function theorem)

Let $f(z,w)$ be a polynomial, $X = V(f) = \{(z,w) \mid f(z,w) = 0\}$.

Let $p = (z_0, w_0) \in X$.

If $\frac{\partial f}{\partial w}(p) \neq 0$, then $\exists$ $U \subset \mathbb{C}$ open, $z_0 \in U$
$V \subset \mathbb{C}^2$ open, $(z_0, w_0) \in V$
and $g: U \to \mathbb{C}$ holomorphic,

such that $X \cap V = \{(z, g(z)) \mid z \in U\}$



That is, near a point where $\frac{\partial f}{\partial w} \neq 0$, $f(z,w) = 0$ defines w as a function of z, and X is locally a graph $w = g(z)$.

Similarly, near a point where $\frac{\partial f}{\partial z} \neq 0$, $f(z,w) = 0$ defines z as a function of w, and X is locally a graph $z = h(w)$

Proof that f nonsingular $\Rightarrow$ $X = V(f)$ is a Riemann surface.

Let $p \in X$. If $\frac{\partial f}{\partial w}(p) \neq 0$, then implicit function theorem says

$X \cap V = \{(z, g(z)) \mid z \in U\}$, open in $\mathbb{C}$

open in X

$\pi_z : X \cap V \to U$ projection to z coord. is a chart.

If $\frac{\partial f}{\partial w}(p) = 0$, then $\frac{\partial f}{\partial z}(p) \neq 0$ (f is nonsingular)

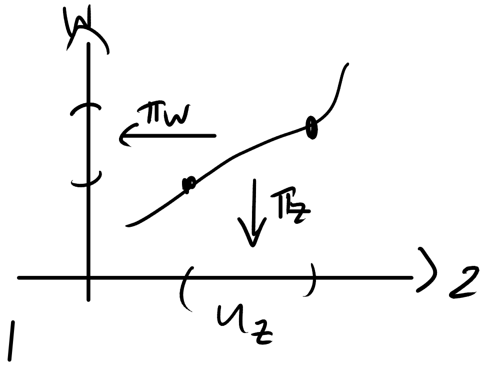
$$X \cap V = \{ (h(w), w) \mid w \in U \}$$

$\pi_w : X \cap V \longrightarrow U$ projection to w coord is a chart.

Compatibility:

$$\begin{tikzcd} X \cap V \arrow[r, "\pi_w"] \arrow[d, "\pi_z"'] & U_w \\ U_z \arrow[ur, "\pi_w \circ \pi_z^{-1}"'] & \end{tikzcd}$$

$\pi_w \circ \pi_z^{-1} = g$ is holomorphic!



QED.

More generally, give any f nonsingular or not, $X = V(f)$ we define

singular locus $X^{sing} := \{ p \in X \mid \frac{\partial f}{\partial w}(p) = 0 \text{ and } \frac{\partial f}{\partial z}(p) = 0 \}$

smooth locus $X^{smooth} := X \setminus X^{sing}$

We have shown <u>X^{smooth} is a Riemann surface.</u> (possibly disconnected)

Def $f(z,w)$ is <u>reducible</u> if $f(z,w) = g(z,w)\, h(z,w)$ where g and h are nonconstant polynomials otherwise f is <u>irreducible</u>.

<u>Theorem</u> If $f(z,w)$ is irreducible, then $X = V(f) \subset \mathbb{C}^2$ is connected.

The proof of this theorem is surprisingly nontrivial - omitted for now.
[Shafarevich's proofs use either a weak form of Riemann-Roch or Noether's Normalization lemma]

<u>Example</u>: If $h(z)$ has distinct roots, then
$f(z,w) = w^2 - h(z)$ is irreducible and nonsingular

$\Rightarrow X = \{(z,w) \mid w^2 = h(z)\}$ is a connected Riemann surf.