The canonical map.

X - compact Riemann Surface

ω a meromorphic 1-form

$K = \mathrm{div}(\omega)$ a canonical divisor.

Recall $\deg K = 2g-2$ and $\ell(K) = g$.

$g=0$: $|K| = \emptyset$

$g=1$: $K \sim 0$, $|K| = \{0\}$

Prop if $g \geq 1$ then K is base point free.

Pf. Need to show $\ell(K-p) = g-1$.

By RR

$$\ell(K-p) - \ell(p) = \deg(K-p) + 1 - g$$
$$= 2g-3+1-g = g-2$$

On a curve of genus $g \geq 1$, $L(p)$ consists of constant functions, [If $f \in L(p)$ is nonconstant, it gives a deg 1 map $X \to \mathbb{P}$!]

So $\ell(p) = 1$, and $\ell(K-p) = g-1$. ▨

Thus, if $g \geq 1$, ϕ_K is a map $\phi_K : X \to \mathbb{P}^{g-1}$

If $g=1$, $\phi_K : X \to \mathbb{P}^0 = \{pt\}$, which is not interesting.

So in what follows, we usually assume $g \geq 2$.

Another description of ϕ_K: Let $\Omega^1(X) = \{\text{holomorphic 1-forms}\}$

Define map

$$\psi : X \longrightarrow \mathbb{P}^*(\Omega^1(X))$$
$$p \longmapsto \{\omega \in \Omega^1(X) \mid \omega(p) = 0\}$$

that is p maps to the hyperplane of 1-forms that vanish at p.

In suitable coordinates, $\psi = \phi_p$.

Genus 2: $\phi_K : X \to \mathbb{P}^1$ has degree $\deg K = 2$
Thus, the canonical map is a degree 2 map to $\mathbb{P}^1$
Recall from the discussion of branched coverings that a surface is hyperelliptic iff it admits a degree 2 map to $\mathbb{P}^1$!

<u>Prop</u> Every genus 2 R.S. X is isomorphic to a hyperelliptic surface, more specifically a degree 2 branched covering of $\mathbb{P}^1$ branched at 6 points.

$g \geq 3$: $\phi_K : X \to \mathbb{P}^{g-1}$ when is it an embedding?

When is it not an embedding? if $\ell(K-p-q) \neq \ell(K) - 2$ for some $p, q \in X$. Since $\ell(K) - 2 \leq \ell(K-p-q) \leq \ell(K-p) = \ell(K) - 1$
this means $\ell(K-p-q) = \ell(K) - 1 = g - 1$
With Riemann-Roch, this implies

$$\ell(K-p-q) - \ell(p+q) = 2g - 4 + 1 - g = g - 3$$
$$g - 1 - \ell(p+q) = g - 3$$
$$\ell(p+q) = 2$$

If $\ell(p+q) = 2$, there is a nonconstant $f \in L(p+q)$
f may only have simple poles at p and q, it cannot have only a single simple pole since $g > 0$, so f has simple poles at p and q. (If $p = q$, this reasoning shows f has a double pole at p.) Thus $f : X \to \mathbb{P}^1$ has degree 2.
Thus X is isomorphic to a hyperelliptic surface.

<u>Conclusion</u>: If $\phi_K : X \to \mathbb{P}^{g-1}$ is not an embedding, X is hyperelliptic.

<u>Prop</u>: If X is hyperelliptic, ϕ_K is not a embedding.

<u>Pf</u>: Let $\pi : X \to \mathbb{P}^1$ be a degree 2 map.
Think of $\pi \in \mathcal{M}^*(X)$ as a nonconstant meromorphic function.
Let $\operatorname{div}_\infty(\pi) = p+q$ be the divisor of poles.
Thus $\pi \in L(p+q)$. Since constants are in $L(p+q)$, we find $\ell(p+q) \geq 2$. Since $\ell(p+q) \leq \ell(p)+1$, and $\ell(p)=1$
we find $\ell(p+q)=2$.
Thus $\ell(K-p-q) = 2g-4+1-g+\ell(p+q) = g-1$
so ϕ_K is not an embedding.

Thus for surfaces of genus $g \geq 3$, we have a <u>dichotomy</u>:
Exactly one of the following holds:
(a) $\phi_K : X \to \mathbb{P}^{g-1}$ is an embedding.
(b) X is hyperelliptic.

If (a), then $\phi_K : X \to \mathbb{P}^{g-1}$ embeds X as a projective curve of degree $2g-2$.

E.g. if $g=3$ and X is not hyperelliptic, $\phi_K : X \to \mathbb{P}^2$ as a curve of degree 4. Thus

> If $g=3$, X is isomorphic to a hyperelliptic curve or to a plane quartic.

If X is hyperelliptic, we can understand the canonical map.
Represent X as the hyperelliptic curve constructed from the equation $y^2 = h(x)$ where h has degree $2g+1$ or $2g+2$ and distinct roots.

By homework $p(x)\frac{dx}{y}$ extends to a holomorphic 1-form on X

iff $p(x)$ is a polynomial of degree $\le g-1$.

So $\frac{dx}{y}, x\frac{dx}{y}, x^2\frac{dx}{y}, \dots, x^{g-1}\frac{dx}{y}$ is a basis of $\Omega^1(X)$

Let $K = \mathrm{div}\left(\frac{dx}{y}\right)$. Then $L(K) = \langle 1, x, x^2, \dots, x^{g-1}\rangle$

In these coords, The canonical map $\phi_K : X \to \mathbb{P}^{g-1}$

$$\phi_K(x,y) = [1 : x : x^2 : \cdots : x^{g-1}]$$

Thus ϕ_K factors through the hyperelliptic projection $\pi : X \to \mathbb{P}^1$
$(x,y) \to [1:x]$

The map $\mathbb{P}^1 \to \mathbb{P}^n$
$[s:t] \to [s^n : s^{n-1}t : \cdots : t^n]$ is called a Veronese map.
(the image is the rational normal curve.)

Thus

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[rr, bend right, "\phi_K"'] & \mathbb{P}^1 \arrow[r, "\nu"] & \mathbb{P}^{g-1} \end{tikzcd}$$

$\phi_K = \nu \circ \pi$
where π is hyperelliptic branched covering and ν is the Veronese map.