Riemann–Roch theorem

Let X be a compact Riemann surface. Let $K = \mathrm{div}(\omega)$ be a canonical divisor on X ($\omega \in \mathcal{M}^{(1)}(X)$ some meromorphic 1-form)
For $D \in \mathrm{Div}(X)$ let $L(D) = \{ f \in \mathcal{M}(X) \mid \mathrm{div}(f) + D \geq 0 \}$

$$\ell(D) = \dim L(D)$$

Theorem (Riemann–Roch + Serre duality)
With notation as above, we have

$$\boxed{\ell(D) - \ell(K-D) = \deg D + 1 - g}$$

where $D \in \mathrm{Div}(X)$ is any divisor and g is the genus of X.

This is a powerful statement: For instance it implies, by taking $D=0$

$$\ell(0) - \ell(K) = 0 + 1 - g$$

but $\ell(0) = 1$ so

$$1 - \ell(K) = 1 - g$$
$$\ell(K) = g.$$

Thus $L(K)$ is g-dimensional. But $L(K) \cong \Omega^1(X)$, the space of holomorphic 1-forms. So we have

$$\boxed{\dim \Omega^1(X) = g}$$

Now taking $D = K$, we have

$$\ell(K) - \ell(K-K) = \deg K + 1 - g$$
$$g \quad - \quad 1 \quad = \deg K + 1 - g$$

$$\boxed{\deg K = 2g - 2}$$

We already knew this, but it's neat to rederive it.

Recall the useful observation:

$$\boxed{\deg D < 0 \Rightarrow \ell(D) = 0}$$

Recall also that the complete linear system $|D|$ defines an embedding iff
$\forall p, q \in X$, $\ell(D-p-q) = \ell(D) - 2$

Prop: If $\deg(D) \geq 2g+1$, then $|D|$ defines an embedding.

Proof:

$$\begin{array}{ccc} \deg(D) \geq 2g+1 & \Longrightarrow & \deg(D-p-q) \geq 2g-1 \\ \Downarrow & & \Downarrow \\ \deg(K-D) \leq -3 & & \deg(K-(D-p-q)) \leq -1 \\ \Downarrow & & \Downarrow \\ \ell(K-D) = 0 & & \ell(K-(D-p-q)) = 0 \end{array}$$

By RR: $\ell(D) - \cancelto{0}{\ell(K-D)} = \deg D + 1 - g$

$\ell(D) = \deg D + 1 - g$

$$\ell(D-p-q) - \cancelto{0}{\ell(K-(D-p-q))} = \deg(D-p-q) + 1 - g$$

$$\ell(D-p-q) = \deg D + 1 - g - 2 = \ell(D) - 2 \quad \blacksquare$$

This in particular implies that X can be embedded in to $\mathbb{P}^n$ for some sufficiently large n. So every compact Riemann surface is a projective curve.

Genus zero Riemann surfaces. Suppose $g = 0$. Thus $\deg K = -2$ and $\ell(D) - \ell(K-D) = \deg D + 1$ for any divisor D.

We can solve the Riemann Roch problem completely.

If $\deg D \geq -1$ then $\deg(K-D) = -2 - \deg D \leq -1$
so $\ell(K-D) = 0$ and we have $\ell(D) = \deg D + 1$
Also if $\deg D < 0$ we have $\ell(D) = 0$. Thus

$$\boxed{\text{If } g=0, \quad \ell(D) = \begin{cases} \deg D + 1, & \text{if } \deg D \geq -1 \\ 0, & \text{if } \deg D < 0 \end{cases}}$$

As an example, consider the divisor $D = p$, where $p \in X$ is a point.
then $\ell(p) = 2$ $\quad \ell(p-q-r) = 0 = \ell(p) - 2$ so
$|p|$ is a basepoint free and defines an embedding.

$$\phi_{|p|} : X \to \mathbb{P}^1$$

Since X carries a 1-1 holomorphic map to $\mathbb{P}^1$, it is isomorphic to $\mathbb{P}^1$

Cor: If $g=0$, $X \cong \mathbb{P}^1 \cong \hat{\mathbb{C}}$.

So consider $X = \hat{\mathbb{C}}$: Two divisors are linearly equivalent if they have the same degree. If $\deg D = n$, then $D \sim n \cdot [\infty]$.

$L(n \cdot [\infty]) = \{ f(z) \mid f \text{ is polynomial of degree } \leq n \}$

so $\ell(n \cdot [\infty]) = n+1$.

$|n \cdot [\infty]| = \{ E \mid E \geq 0 \quad \deg E = n \}$

Now observe that $\{E \mid E \geq 0, \deg E = n\}$ is the same as the set of configurations of n unordered points on $\hat{\mathbb{C}}$

$$|n \cdot [\infty]| \cong \mathrm{Sym}^n \hat{\mathbb{C}} = \underbrace{\hat{\mathbb{C}} \times \cdots \times \hat{\mathbb{C}}}_{n} / S_n$$

On the other hand $\left|n\cdot[\infty]\right| \cong \mathbb{P}(L(n\cdot[\infty])) \cong \mathbb{P}^n$

So we have shown $\mathrm{Sym}^n \hat{\mathbb{C}} = \mathrm{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$!

<u>Genus 1</u>: $\deg K = 2g-2 = 0$.

The Riemann–Roch formula becomes:

$$\ell(D) - \ell(K-D) = \deg D$$

If $\deg D > 0$ then $\deg(K-D) < 0$, so $\ell(K-D) = 0$

We have

$$\text{If } g=1, \quad \ell(D) = \begin{cases} \deg D & \text{if } \deg D > 0 \\ ? & \text{if } \deg D = 0 \\ 0 & \text{if } \deg D < 0 \end{cases}$$

We still need to fill in the ?, but in this case the answer depends on D, not only its degree. Also it is <u>no longer true</u> that all divisors of the same degree are linearly equivalent.

If $\deg D = 0$, and $E \in |D|$, then $E \geq 0$ and $E \sim D$, so $\deg E = 0$

Thus $\deg E = 0$ and $E \geq 0$, but this implies $E = 0$. Thus $D \sim 0$.

If $D \sim 0$, $\ell(D) = \ell(0) = 1$. otherwise $\ell(D) = 0$.

$$\text{If } g=1, \quad \ell(D) = \begin{cases} \deg D & \text{if } \deg D > 0 \\ 1 & \text{if } \deg D = 0 \text{ and } D \sim 0 \\ 0 & \text{if } \deg D = 0 \text{ and } D \not\sim 0 \\ 0 & \text{if } \deg D < 0 \end{cases}$$

This solves the problem, but requires us to understand the conditions $D \sim 0$ $D \not\sim 0$ for degree 0 divisors.

The divisor $D = p$ is not basepoint free, since $L(p) = 1$ but
$L(p-p) = l(0) = 1 \neq L(p) - 1$. So indeed p is a basepoint.
From this we see that $|p| = \{p\}$, i.e., there is only one nonnegative divisor linearly equivalent to p, namely p itself. Thus distinct points on X are <u>not</u> linearly equivalent.
Thus, if $p \neq q$, then $p - q$ is a divisor of degree 0 that is not ~ 0.

The divisor $D = 2p$ is basepoint free $L(2p) = 2$ and $L(2p-q) = 1$
It defines a map $\phi_{2p} : X \to \mathbb{P}^* L(2p) \cong \mathbb{P}^1$ The degree of this map is $\deg D = 2$. Thus X admits a degree 2 map to $\mathbb{P}^1$, and hence is hyperelliptic. Hence

$$\boxed{\text{If } g = 1, \quad \text{then } X \text{ is hyperelliptic}}$$

The divisor $D = 3p$ defines an embedding. $l(3p) = 3$ and $l(3p - q - r) = 1$
It defines a map $\phi_{3p} : X \to \mathbb{P}^2$. The degree of the image is 3, in the sense that a general line in $\mathbb{P}^2$ intersects $\phi_{3p}(X)$ in 3 points.

$$\boxed{\text{If } g = 1, \quad \text{then } X \text{ is isomorphic to a cubic plane curve.}}$$