Linear systems $\longrightarrow$ maps $X \to \mathbb{P}^n$

Let ∂ be a linear system of divisors on X. Recall that this means that there is a divisor $D \in \mathrm{Div}(X)$ such that

$$\partial \subseteq |D| = \{ E \mid E \geqslant 0 , E \sim D \}$$

And furthermore, there is a subspace $V_\partial \subseteq L(D)$ such that, under the isomorphism

$$|D| \cong \mathbb{P}(L(D)) = (L(D) \setminus \{0\}) / \mathbb{C}^*,$$

we have

$$\partial \cong \mathbb{P}(V_\partial) = (V_\partial \setminus \{0\}) / \mathbb{C}^* .$$

We can then construct a map $\phi : X \to \mathbb{P}^r$ where $r = \dim \partial$
$r+1 = \dim V_\partial$

Namely, choose a basis $f_0, \dots, f_r$ for V_∂, and let $\phi = \phi_f$

$$\phi(x) = [f_0(x) : \cdots : f_r(x)]$$

[We should understand the dependence of this map on the choice $V_\partial \subset L(D)$, but let's set that aside for now.]

Now if we start with $\partial \subseteq |D|$, and we form ϕ, then we take $|\phi|$. Do we get ∂ back? Not quite, but close.

There is another property of the linear system $|\phi|$ for $\phi : X \to \mathbb{P}^r$ that we have not yet used:

<u>Def</u> Let $\mathfrak{d}$ be a linear system on X, and let $p \in X$.

* p is a <u>base point</u> of $\mathfrak{d}$ if $\forall E \in \mathfrak{d}$, $E \geq p$, that is, if every divisor in $\mathfrak{d}$ contains p in its support.
* $\mathfrak{d}$ is called <u>base-point-free</u> if it has no base points

<u>Prop</u>: Let $\phi : X \to \mathbb{P}^r$ be a hol. map. The linear system $|\phi|$ is base-point-free.

<u>Proof</u>: Let $p \in X$, $\phi = [f_0 : \cdots : f_r]$ for $f_i \in \mathcal{M}(X)$

$D = -\min\{\operatorname{div}(f_i)\}$. Let $D(p) = -\min\{\operatorname{ord}_p(f_i)\} = -k$

Thus $\operatorname{ord}_p(f_i) \geq k$ for all i, and there exists j such that $\operatorname{ord}_p(f_j) = k$.

Consider $E = \operatorname{div}(f_j) + D \in |\phi|$.

Then $E(p) = \operatorname{ord}_p(f_j) + D(p) = k - k = 0$. So $E \not\geq p$, and p is not a base point of $|\phi|$. Since p was arbitrary, $|\phi|$ is base point free. $\blacksquare$

<u>Prop</u> Let $\mathfrak{d} \subseteq |D|$ be linear system corresponding to $V \subseteq L(D)$. Let $p \in X$. Then p is a base point of $\mathfrak{d}$ iff $V \subseteq L(D-p)$

* In particular, p is a base-point of $|D|$ iff $L(D-p) = L(D)$ iff $\dim L(D-p) = \dim L(D)$. (assuming X compact)
* $|D|$ is base point free iff $\forall p \in X$, $\dim L(D-p) = \dim L(D) - 1$.

There is also a special property of maps $\phi : X \to \mathbb{P}^r$ that arise from linear systems. Let $\mathbb{P} = \mathbb{P}(V) = (V \setminus \{0\})/\mathbb{C}^*$ be a projective space. A subset $S \subseteq \mathbb{P}$ <u>spans</u> $\mathbb{P}$ if the smallest <u>linear</u> subspace of $\mathbb{P}$ containing S is all of S. This happens if $\pi^{-1}(S)$ spans V, where $\pi : V \setminus \{0\} \to \mathbb{P}(V)$ is the quotient map.

<u>Def</u>: A map $X \to \mathbb{P}^r$ is <u>nondegenerate</u> if its image spans $\mathbb{P}^r$.

<u>Prop</u>: Let $f_0, \ldots, f_r \in \mathcal{M}(X)$ be functions that are linearly independent over $\mathbb{C}$. Then $\phi_f = [f_0 : \cdots : f_r] : X \to \mathbb{P}^r$ is a nondegenerate map.

<u>Pf</u>: If the map were degenerate, there would be a linear relation
$$a_0 f_0 + \cdots + a_r f_r = 0$$
$\blacksquare$

Now we come to the main statement relating linear systems to maps to $\mathbb{P}^r$:

<u>Theorem</u>: There is a 1-1 correspondence.

$$\left\{\begin{matrix}\text{base point free}\\ \text{linear systems } \partial\\ \text{of dimension } r\\ \text{on } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{non degenerate}\\ \text{holomorphic maps}\\ \phi : X \to \mathbb{P}^r\end{matrix}\right\} \Big/ \left\{\begin{matrix}\text{linear changes of}\\ \text{coordinates on } \mathbb{P}^r\end{matrix}\right\}$$

$$\partial \longmapsto \phi = [f_0 : \cdots : f_r]$$
with basis $f_0, \ldots, f_r$ for $V_\partial \subseteq L(D)$

$$|\phi| \longleftarrow \phi$$

We now want to understand the maps $\phi_D : X \to \mathbb{P}^r$ associated to the complete linear system $|D|$. The main question is whether ϕ_D is an embedding.

<u>Recall</u>: $|D|$ is base point free $\Leftrightarrow$ $\forall p \in X$, $\dim L(D-p) = \dim L(D) - 1$

Another description of the map $\phi_D : X \to \mathbb{P}^r$ associated to $|D|$.

Recall: V a vectorspace: $\mathbb{P}(V) = \{$dimension 1 subspaces in $V\}$

$\mathbb{P}^*(V) = \{$codimension 1 subspaces in $V\}$

$V^* = \{$linear maps $V \to \mathbb{C}\}$

$\mathbb{P}(V^*) = \{$dimension 1 subspaces in $V^*\}$

There is a natural isomorphism $\mathbb{P}(V^*) \longrightarrow \mathbb{P}^*(V)$

$$\lambda \in V^* \setminus \{0\} \longmapsto \ker \lambda \subset V$$

So $\mathbb{P}^*(V)$ is a projective space.

Given a base-point-free divisor D, we will construct a map

$$\psi : X \longrightarrow \mathbb{P}^*(L(D))$$

We need to associate to $p \in X$ a codimension 1 subspace $\psi(p) \subseteq L(D)$

Let $\psi(p) = L(D-p) \subseteq L(D)$

This has codimension at most 1, and it has codimension 0 iff p is a basepoint of $|D|$. Since D is assumed base-point-free, $\psi(p)$ has codim 1.

Prop Under a suitable identification $\mathbb{P}^*(L(D)) \cong \mathbb{P}(L(D)^*) \cong \mathbb{P}^r$, ψ is precisely the map $\phi_D : X \to \mathbb{P}^r$ associated to $|D|$.

One can also think of $\mathbb{P}^*(L(D))$ as the set of codim 1 subspaces in $\mathbb{P}(L(D)) \cong |D|$, denoted $|D|^*$.

So $\psi : X \to |D|^*$

Now assuming $|D|$ is base point free, when is ψ or ϕ_D an embedding?

$$\psi(p) = \psi(q) \Leftrightarrow L(D-p) = L(D-q) \subseteq L(D)$$

Generally: $L(D-p) \cap L(D-q) = L(D-p-q)$

So $L(D-p) = L(D-q)$ iff $L(D-p-q) = L(D-p) = L(D-q)$.

Thus $\psi(p) \neq \psi(q)$ iff in the diagram

$$\begin{array}{ccc} & L(D) & \\ \cup\!\!\!/ & & \cup\!\!\!/ \\ L(D-p) & & L(D-q) \\ \cup\!\!\!/ & & \cup\!\!\!/ \\ & L(D-p-q) & \end{array}$$

all inclusions are proper

iff $\dim L(D-p-q) = \dim L(D) - 2$.

<u>Prop</u>: Assume D is base point free $(\forall p \ \dim L(D-p) = \dim L(D) - 1)$

ψ is 1-1 if $\forall p, q$ distinct $\dim L(D-p-q) = \dim L(D) - 2$

ψ is a holomorphic embedding if additionally

$\dim L(D-2p) = \dim L(D) - 2$.

<u>Comment</u>: The condition $\dim L(D-2p) = \dim L(D) - 2$ corresponds to the requirement that the map $\psi : X \to \mathbb{P}^*(L(D))$ be an immersion.

<u>Example</u> Rational normal curves. $X = \hat{\mathbb{C}}$, $D = n \cdot [\infty]$ $n \geq 1$

$L(D) = \{ f \mid \mathrm{div}(f) + n[\infty] \geq 0 \} = \{ f(z) \mid f$ is polynomial of degree $\leq n \}$

$\dim L(D) = \deg(D) + 1$ for all divisors $D \geq 0$.

Thus $\dim L(D) = n+1$

$\forall p \quad \dim L(D-p) = n$ so D is base point free,

$\forall p, q \quad \dim L(D-p-q) = n-1$ so ϕ_D is an embedding.

The image of $\phi_D : \hat{\mathbb{C}} \to \mathbb{P}^n$ is called a <u>rational normal curve</u>.

Choosing a basis for $L(D)$, say $1, z, \ldots, z^n \in L(D)$
we have the map

$$\begin{aligned} \phi_D : X &\to \mathbb{P}^n \\ z &\mapsto [1 : z : \cdots : z^n] \\ \infty &\mapsto [0 : 0 : \cdots : 1] \end{aligned}$$

In homogeneous coordinates $[x:y]$ on $\mathbb{P}^1 \cong \hat{\mathbb{C}}$
$\left(\frac{y}{x} = z\right)$

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^n \\ [x:y] &\longmapsto \left[1 : \frac{y}{x} : \left(\frac{y}{x}\right)^2 : \cdots : \left(\frac{y}{x}\right)^n\right] = [x^n : x^{n-1}y : \cdots : y^n] \end{aligned}$$

$n = 1$: $\quad \mathbb{P}^1 \to \mathbb{P}^1$, $\quad [x:y] \mapsto [x:y]$ $\quad$ identity map.

$n = 2$: $\quad \mathbb{P}^1 \to \mathbb{P}^2$, $\quad [x:y] \mapsto [x^2 : xy : y^2]$ $\quad$ image is $\{[z_0 : z_1 : z_2] \mid z_1^2 = z_0 z_2\}$
smooth conic.

$n = 3$: $\quad \mathbb{P}^1 \to \mathbb{P}^3$, $\quad [x:y] \mapsto [x^3 : x^2y : xy^2 : y^3]$ $\quad$ twisted cubic curve.