Divisors and Maps to $\mathbb{P}^n$

<u>Def</u> A map $\phi : X \to \mathbb{P}^n$ is <u>holomorphic at $p \in X$</u> if
There exist holomorphic functions $g_0, g_1, \dots, g_n$ defined in some open set $U \subset X$, $p \in U$, which do not all vanish simultaneously, such that

$$\phi(x) = [g_0(x) : g_1(x) : \dots : g_n(x)] \text{ for } x \in U.$$

ϕ is <u>holomorphic</u> if it is holomorphic at every point.

The functions $g_i(x)$ are <u>not</u> determined by the map ϕ, because of the scaling ambiguity of homogenous coords.
if $f(x)$ is a holomorphic function in U, nonvanishing

$$[f(x)g_0(x) : \dots : f(x)g_n(x)] = [g_0(x) : \dots : g_n(x)]$$

We can also define maps using meromorphic functions

Choose $n+1$ meromorphic fns $f = (f_0, \dots, f_n) \in \mathcal{M}(X)^{n+1}$
not all identically zero.

Let's try to define $\phi_f : X \to \mathbb{P}^n$

$$\phi_f(x) \longrightarrow [f_0(x) : \dots : f_n(x)]$$

This obviously is ok a $p \in X$ if $\begin{cases} p \text{ is not a pole of any } f_i \text{ and} \\ p \text{ is not a zero of all } f_i \end{cases}$

<u>Lemma</u>:
In fact, this map can be extended uniquely to all of X.

<u>Proof</u> Fix $p \in X$. Let $m = \min\{\mathrm{ord}_p(f_i)\}$

If $m=0$: All f_i are holomorphic at p, and at least one is nonzero $\Longrightarrow$ OK

If $m>0$: All f_i are holomorphic at p, but the all vanish.

If $m<0$: Some f_i has a pole at p.

Let z be a local coordinate at $p \in X$

In either case, modify $f=(f_0,\dots,f_n)$ to $z^{-m}f=(z^{-m}f_0,\dots,z^{-m}f_n)$

Then $\min\{\mathrm{ord}_p(z^{-m}f_i)\} = m-m=0$,
so all $z^{-m}f_i$ are holomorphic at $p \leftrightarrow (z=0)$
and at least one is nonzero.
On the other hand

$$\phi_f(z) = [f_0(z):\cdots:f_n(z)] = [z^{-m}f_0(z):\cdots:z^{-m}f_n(z)] = \phi_{z^{-m}f}$$

at all points where both sides make sence
and the right hand side makes sense at p!

So we define ϕ_f to be $\phi_{z^{-m}f}$ near p.

<u>Proposition</u>: Let $\phi: X \to \mathbb{P}^n$ be a holomorphic map. Then there is an $(n+1)$ tuple $f=(f_0,\dots,f_n) \in \mathcal{M}(X)^{n+1}$ such that

$$\phi(x) = \phi_f(x) = [f_0(x):\cdots:f_n(x)]$$ as in the lemma.

Two $(n+1)$ tuples $f, g \in \mathcal{M}(X)^{n+1}$ define the same map if and only if there is $\lambda \in \mathcal{M}^*(X)$ such that

$$f = \lambda g \quad (f_i = \lambda g_i \ , i = 0, \dots, n)$$

Proof: For existence let $\phi : X \to \mathbb{P}^n$ be a holomorphic map. After reordering coordinates, we may assume that x_0 does not vanish identically on X.

Set $f_i = \frac{x_i}{x_0}$, $i = 0, \dots, n$ (so $f_0 = 1$)

then f_i is a meromorphic function on X and $\forall p \in X$

$$\phi(p) = [x_0 : \dots : x_n] = \left[1 : \frac{x_1}{x_0} : \dots \frac{x_n}{x_0}\right] = \underbrace{[f_0 : \dots : f_n]}_{= \phi_f}.$$

the other parts are straightforward ☒

Corollary: $\{\text{maps } X \to \mathbb{P}^n\} \leftrightarrow \left\{ \begin{array}{l} 1\text{-dimensional } \mathcal{M}(X)\text{-linear subspaces} \\ \text{of the } \mathcal{M}(X)\text{-vectorspace } \mathcal{M}(X)^{n+1} \end{array} \right\}$

$$\| $$

$\mathbb{P}^n_{\mathcal{M}(X)}$ proj space over the field $\mathcal{M}(X)$.

Holo. maps $\longrightarrow$ linear systems. Let $\phi : X \to \mathbb{P}^n$ be a holomap.

Write $\phi = \phi_f$ for some tuple $(f_0, \dots, f_n) \in \mathcal{M}(X)^{n+1}$

Set $D = -\min\{\operatorname{div}(f_i)\}$, i.e. $D(p) = -\min\{\operatorname{ord}_p(f_i)\}$

Then $\mathrm{div}(f_i) + D \geq 0$ by construction, so $f_i \in L(D)$ $\forall i$

Let $V_f = \mathrm{span}(f_0, \ldots, f_n) \subseteq L(D)$ (this may be a proper subspace)

Now set $|\phi| = \{ \mathrm{div}(g) + D \mid g \in V_f \}$

Every divisor in $|\phi|$ is nonnegative and linearly equivalent to D, so

$$|\phi| \subseteq |D| \leftarrow \text{the complete linear system assoc. to } D.$$

$$\Big(\text{under the iso } |D| \cong \mathbb{P}(L(D)), \quad |\phi| = \mathbb{P}(V_f) \subseteq \mathbb{P}(L(D)) \Big)$$

Lemma $|\phi| \subseteq \mathrm{Div}(X)$ is well-defined, independent of the tuple $f = (f_0, \ldots, f_n)$ used in the construction.

Proof: suppose g and f define the same map ϕ. then $g = \lambda f$ for some $\lambda \in \mathcal{M}^*(X)$.

$$D_g = -\min\{\mathrm{div}(g_i)\} = -\min\{\mathrm{div}(\lambda f_i)\} = -\min\{\mathrm{div}(\lambda) + \mathrm{div}(f_i)\}$$

$$= -\mathrm{div}(\lambda) - \min\{\mathrm{div}(f_i)\} = -\mathrm{div}(\lambda) + D_f$$

Thus $D_g \sim D_f$ and so $|D_g| = |D_f|$

Now $Q \in |\phi_g| \Leftrightarrow Q = \mathrm{div}(\sum a_i g_i) + D_g =$

$$= \mathrm{div}(\textstyle\sum a_i \lambda f_i) + D_g = \mathrm{div}(\lambda) + \mathrm{div}(\sum a_i f_i) + D_g$$

$$= \mathrm{div}(\textstyle\sum a_i f_i) + \underbrace{\mathrm{div}(\lambda) + D_g}_{D_f} = \mathrm{div}(\sum a_i f_i) + D_f$$

$\Leftrightarrow Q \in |\phi_f|$.