Spaces of meromorphic functions and forms associated to a divisor

<u>Convention</u>: • if f is identically zero in a neighborhood of p, define $\operatorname{ord}_p(f) = \infty$

• $\infty > n$ for any finite integer n.

<u>Def</u> Let $D \in \operatorname{Div}(X)$ be a divisor, $f \in \mathcal{M}(X)$ a meromorphic function. We say <u>f has poles bounded by D</u> if

$$\operatorname{div}(f) \geq -D \iff \operatorname{div}(f) + D \geq 0 \iff \operatorname{ord}_p(f) + D(p) \geq 0 \quad \forall p \in X$$

Let $L(D) = \{ f \in \mathcal{M}(X) \mid \operatorname{div}(f) \geq -D \}$.

Remark: with our conventions, $0 \in L(D)$. Also, the property $\operatorname{ord}_p(f+g) \geq \min\{\operatorname{ord}_p(f), \operatorname{ord}_p(g)\}$ implies that

if $f, g \in L(D)$, then $f+g \in L(D)$

So $L(D)$ is a complex vector space.

<u>Observations</u>: if $D_1 \leq D_2$ then $L(D_1) \subseteq L(D_2)$

Since $\operatorname{div}(f) \geq 0 \iff f$ is holomorphic, $L(0) = \mathcal{O}(X) = \{\text{hol. functions on } X\}$

if X is compact $L(0) \cong \mathbb{C}$

<u>Lemma</u> If X is compact and $D \in \operatorname{Div}(X)$ such that $\deg(D) < 0$ then $L(D) = \{0\}$

<u>Proof</u> suppose $f \in L(D)$ not identically zero. Then $\operatorname{div}(f) + D \geq 0$

But then $\deg(\operatorname{div}(f) + D) \geq 0$.

But $\deg(\operatorname{div}(f) + D) = \deg(\operatorname{div}(f)) + \deg(D) = 0 + \deg(D) < 0$ ↯

Complete linear systems of divisors

Def Given $D \in \mathrm{Div}(X)$, let $|D| = \{E \in \mathrm{Div}(X) \mid E \geq 0, E \sim D\}$

Note that $D_1 \sim D_2 \implies |D_1| = |D_2|$

If X is compact and $\deg(D) < 0$, then $|D| = \emptyset$.

Now, given a complex vectorspace V, denote by $\mathbb{P}(V)$ the set of complex 1-dim'l subspaces of V, called the projectivization.

Lemma X compact. The map
$$S: L(D)\setminus\{0\} \to |D|$$
$$f \mapsto \mathrm{div}(f) + D$$
induces a bijection $\bar{S}: \mathbb{P}(L(D)) \to |D|$

Proof: $\bar{S}$ is well defined: Suppose $f_1, f_2 \in L(D)$ lie in the same 1-dim subspace. Then $f_1 = cf_2$, c constant
so $\mathrm{div}(f_1) = \mathrm{div}(c) + \mathrm{div}(f_2) = \mathrm{div}(f_2)$.
It is clear that $\mathrm{div}(f) + D \in |D|$, since $\mathrm{div}(f) + D \sim D$ and $\mathrm{div}(f) + D \geq 0$ by definition.
To see that $\bar{S}$ is surjective: Let $E \in |D|$, so $E \geq 0$ and $E \sim D$ then we can write $E = D + \mathrm{div}(f)$. Since $E \geq 0$, $f \in L(D)$
To see that $\bar{S}$ is injective, suppose $\mathrm{div}(f_1) + D = \mathrm{div}(f_2) + D$
so $\mathrm{div}(f_1) = \mathrm{div}(f_2)$ $\mathrm{div}(f_1/f_2) = 0$ so f_1/f_2 is holomorphic hence constant since X is compact. Thus f_1 and f_2 lie in the same 1-dim'l subspace of $L(D)$ ∎

Def A linear system is a set of divisors of the form $\bar{S}(V) \subset |D|$ where $V \subset L(D)$ is a linear subspace.
The dimension $\dim \bar{S}(V) = \dim V - 1$

Linear equivalence and isomorphisms of $L(D)$'s.

Prop Suppose D_1 and D_2 are linearly equivalent : $D_1 = D_2 + \mathrm{div}(h)$
Then $\mu_h : L(D_1) \longrightarrow L(D_2)$ is an isomorphism
$f \longmapsto hf$

In particular, if $D_1 \sim D_2$, $\dim L(D_1) = \dim L(D_2)$

Pf: $f \in L(D_1) \Leftrightarrow \mathrm{div}(f) + D_1 \geq 0 \Leftrightarrow \mathrm{div}(f) + D_2 + \mathrm{div}(h) \geq 0$
$\Leftrightarrow \mathrm{div}(hf) + D_2 \geq 0 \Leftrightarrow hf \in L(D_2)$

There is an analogous story for 1-forms.

Def $L^{(1)}(D) = \{ \omega \in \mathcal{M}^{(1)}(X) \mid \mathrm{div}(\omega) + D \geq 0 \}$

$L^{(1)}(D)$ is a complex vector space, and $L^{(1)}(0) = \Omega^1(X)$ is the space of holomorphic 1-forms.

Prop Suppose $D_1 \sim D_2$, so $D_1 = D_2 + \mathrm{div}(h)$ then
$\mu_h : L^{(1)}(D_1) \rightarrow L^{(1)}(D_2)$ is an isomorphism
$\omega \longmapsto h\omega$

Prop Fix a canonical divisor $K = \mathrm{div}(\omega)$ for some $\omega \in \mathcal{M}^{(1)*}(X)$
Then the map of multiplication by ω defines a map
$\mu_\omega : L(D+K) \rightarrow L^{(1)}(D)$
$f \longmapsto f\omega$

that is an isomorphism.

Proof $f \in L(D+K) \Leftrightarrow \mathrm{div}(f) + D + K = \mathrm{div}(f) + D + \mathrm{div}(\omega)) \geq 0$
$\Leftrightarrow \mathrm{div}(f\omega) + D \geq 0 \Leftrightarrow f\omega \in L^{(1)}(D)$ ∎

The Riemann-Roch problem is to compute the dimension of $L(D)$ for various divisors $D \in \mathrm{Div}(X)$. The Riemann-Roch theorem and Serre duality give strong constraints, although they do not solve it explicitly. The first thing to do is prove that it has a finite answer, if X is compact.

Lemma: let X be a R.S. let $D \in \mathrm{Div}(X)$. let $p \in X$. We have $L(D-p) \subset L(D)$ and the quotient $L(D)/L(D-p)$ has dimension 0 or 1.

Proof suppose $D(p) = n_p$ is the coefficient of $p \in D$.
Observe that $f \in L(D) \Rightarrow \mathrm{ord}_p(f) + n_p \geq 0$.
Taking the Laurent expansion of f at p, we find it looks like $f = \sum_{i=-n_p}^{\infty} a_i z^i$. Define a map $L(D) \longrightarrow \mathbb{C}$
by sending f to a_{-n_p} in this expansion. The kernel of this map is precisely $L(D-p)$, since if $f \in L(D)$, f will be in $L(D-p)$ iff additionally $a_{-n_p} = 0$, so $\mathrm{ord}_p(f) \geq -n_p + 1 \Leftrightarrow \mathrm{ord}_p(f) + n_p - 1 \geq 0$.

If $L(D) \rightarrow \mathbb{C}$ is surjective $L(D)/L(D-p) \cong \mathbb{C}$
If $L(D) \rightarrow \mathbb{C}$ is not surjective, it is zero map, so $L(D)/L(D-p) = 0$ ∎

Prop: let X be a compact R.S. then $L(D)$ is finite dimensional.
More specifically, if $D = P - N$ where $P \geq 0$ and $N \geq 0$ with disjoint supports, then $\dim L(D) \leq 1 + \deg(P)$.
In particular, if $D \geq 0$, then $\dim L(D) \leq 1 + \deg(D)$

Proof Since $D \leq P$, $L(D) \subseteq L(P)$ and $\dim L(D) \leq \dim L(P)$.
Induction on $\deg P$. If $\deg P = 0$ then $P = 0$ and $L(0) = \mathbb{C}$,
so $\dim L(D) = 1$

Suppose we know that for all $P \geq 0$, $\deg P = k-1$,
we have $\dim L(P) \leq 1 + \deg(P) = k$

Let $P \geq 0$ have $\deg P = k$. Choose q in the support of P, and consider $P-q$. This has $P-q \geq 0$ and $\deg(P-q) = k-1$, so by the induction hypothesis,

$$\dim L(P-q) \leq 1 + \deg(P-q) = k$$

Now by the lemma, $\dim L(P)/L(P-q) \leq 1$
so $\dim L(P)$ is at most 1 greater than $\dim L(P-q)$
$\Rightarrow \dim L(P) \leq \dim L(P-q) + 1 \leq k+1$. $\blacksquare$

<u>Cor</u> Let X be compact R.S. Then $\dim L^{(1)}(D) < \infty$.

<u>Proof</u> $L^{(1)}(D) \simeq L(D+K)$, where K is a canonical divisor.