Linear Equivalence

<u>Def</u>: Divisors $D_1, D_2 \in Div(X)$ are *linearly equivalent*, $D_1 \sim D_2$, if $D_1 - D_2 \in PDiv(X)$, that is, there is a meromorphic function f such that $D_1 - D_2 = div(f)$.

<u>Lemma</u> (a) linear equivalence is an equivalence relation indeed.
(b) $D \sim 0 \iff D \in PDiv(X) \iff D = div(f)$ for some $f \in \mathcal{M}^*(X)$
(c) if $D_1 \sim D_2$ then $deg(D_1) = deg(D_2)$

In fact, the set of linear equivalence classes is the coset space of $PDiv(X)$ in $Div(X)$ or in other words the quotient group $Cl(X) = \dfrac{Div(X)}{PDiv(X)}$

this is called the <u>divisor class group</u>. $Cl(X)$.

<u>Examples</u> (a) f meromorphic on X, not identically zero, then the divisor of zeros $div_0(f)$ and the divisor of poles $div_\infty(f)$ are linearly equivalent, indeed
$$div_0(f) - div_\infty(f) = div(f) \in PDiv(X).$$

(b) any two canonical divisors are linearly equivalent: $\omega_2 = f\omega_1$
$$div(\omega_2) - div(\omega_1) = div(f) \in PDiv(X)$$

(c) $X = \hat{\mathbb{C}}$, then any two points p, q are linearly equivalent: consider
$p, q \neq \infty$: $f = \dfrac{z-p}{z-q}$ $div(f) = p - q$ or $q = \infty$: $f = z - p$ $div(f) = p - \infty$

(d) $F: X \to Y$ hol map. $D_1, D_2 \in Div(Y)$ such that $D_1 \sim D_2$ on Y then $F^*(D_1) \sim F^*(D_2)$ on X.

(e) If $F : X \to \hat{\mathbb{C}}$ is a holomorphic map all inverse images $F^*(\lambda)$ are linearly equivalent

(f) X smooth projective curve in $\mathbb{P}^n$, G_1 and G_2 homog polys of same degree d, then the intersection divisors $\mathrm{div}(G_1)$ and $\mathrm{div}(G_2)$ are linearly equivalent.

Example (e) is what gives the name linear equivalence: we think of $F^*(\lambda)$ $\lambda \in \hat{\mathbb{C}} \cong \mathbb{P}^1$ as a linear family of divisors on X.

Corollary (a) f nonzero meromorphic function on X, then
$$\deg(\mathrm{div}_0(f)) = \deg(\mathrm{div}_\infty(f))$$

(b) Any two canonical divisors have the same degree: $2g-2$

(c) If X is a smooth projective curve, and G_1 and G_2 are homogenes of the same degree, $\deg(\mathrm{div}(G_1)) = \deg(\mathrm{div}(G_2))$

Example of Riemann sphere:

Proposition: A divisor on the Riemann sphere $\hat{\mathbb{C}} \cong \mathbb{P}^1$ is principal iff it has degree 0. That is, $\mathrm{PDiv}(\hat{\mathbb{C}}) = \mathrm{Div}_0(\hat{\mathbb{C}})$

Proof $\mathrm{PDiv} \subset \mathrm{Div}_0$ always. So let $D \in \mathrm{Div}_0(\hat{\mathbb{C}})$. Write
$$D = \sum_{i=1}^{r} e_i \lambda_i + e_\infty \infty \quad \text{for some } \lambda_i \in \mathbb{C}.$$
Since $\deg(D) = 0$, $e_\infty = -\sum_{i=1}^{r} e_i$. Then $D = \mathrm{div}(f)$, where
$$f = \prod_{i=1}^{r} (z - \lambda_i)^{e_i}$$
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Corollary: The divisor class group $Cl(\hat{\mathbb{C}})$ of $\hat{\mathbb{C}} (\cong \mathbb{P}^1)$ is isomorphic to $\mathbb{Z}$.

Proof: $Cl(\hat{\mathbb{C}}) = \dfrac{Div(\hat{\mathbb{C}})}{PDiv(\hat{\mathbb{C}})}$ by def, and $PDiv(\hat{\mathbb{C}}) = Div_0(\hat{\mathbb{C}})$ by prop.

so $Cl(\hat{\mathbb{C}}) = \dfrac{Div(\hat{\mathbb{C}})}{Div_0(\hat{\mathbb{C}})}$. Then we observe there is an exact sequence

$$0 \to Div_0(X) \to Div(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$ $\blacksquare$

In general, there is an exact sequence $0 \to Cl_0(X) \to Cl(X) \xrightarrow{\deg} \mathbb{Z} \to 0$ where $Cl_0(X) = \dfrac{Div_0(X)}{PDiv(X)}$ is the degree 0 divisor class group.

For $X = \hat{\mathbb{C}} \cong \mathbb{P}^1$, $Cl_0(X)$ is trivial. In general it is a large (uncountable) group, isomorphic to a torus of dimension $2g$ $(S^1)^{2g}$. This is the subject of Abel's theorem, which we shall see later.

Degree of a smooth projective curve: $X \subset \mathbb{P}^n$ smooth projective curve.

Def The degree of X in $\mathbb{P}^n$ is the degree of any hyperplane divisor on X, that is $\deg(div(G))$ where $G = a_0x_1 + \cdots + a_nx_n$ is a linear form that does not vanish identically on X.

Prop Let $X \subset \mathbb{P}^2$ be smooth projective curve defined by the equation $F(x,y,z) = 0$. Then the degree of X in the sense just defined equals $\deg(F)$.

Proof: By changing coordinates, WLOG assume $[0:0:1] \notin X$

Consider x, a linear homog poly. We want to compute $\deg(div(x))$

To compute $div(x)$, choose another linear homog poly say y that does not vanish where x does, and compute $ord_p\left(\frac{x}{y}\right)$

at all points $p \in X$ when $x = 0$. thus $\text{div}(x) = \text{div}_0\left(\frac{x}{y}\right)$.

Now consider $h = \frac{x}{y}$ as a map $H : X \to \hat{\mathbb{C}}$

Clearly $\deg(\text{div}_0(h)) = \deg(H)$, the degree of H as a map.

For $\lambda \neq 0$, $H^{-1}(\lambda) = \left\{ [x:y:z] \,\middle|\, \frac{x}{y} = \lambda \text{ and } F(x,y,z) = 0 \right\}$

$$= \{ [\lambda : 1 : w] \mid F(\lambda, 1, w) = 0 \}$$

If λ is not a branch point of H, the equation $F(\lambda, 1, w) = 0$ has $d = \deg(F)$ solutions, each with multiplicity 1, so

$$\deg(\text{div}(x)) = \deg(\text{div}_0(h)) = \deg(H) = \deg(F). \quad \blacksquare$$

<u>Bézout's theorem</u>: Let $X \subset \mathbb{P}^n$ be smooth projective curve of degree d. Let G be a homog polynomial of degree e. that is not identically zero on X. Then

$$\deg(\text{div}(G)) = d \cdot e = \deg(X) \cdot \deg(G)$$

<u>Proof</u> Let H be a linear homog poly defining a hyperplane divisor on X. then $\deg(\text{div}(H)) = d$. since H^e has degree $e = \deg(G)$ we know $\text{div}(H^e) \sim \text{div}(G)$. Thus

$$\deg(\text{div}(G)) = \deg(\text{div}(H^e)) = \deg(e \cdot \text{div}(H)) = e \cdot \deg(\text{div}(H)) = e \cdot d.$$