Divisors

Let X be a compact Riemann surface. A Divisor on X is a formal sum of points $p \in X$. Formally

$$\mathbb{Z}^X = \{\text{all functions } X \to \mathbb{Z}\}$$

given $f : X \to \mathbb{Z}$, $\operatorname{supp}(f) = \{p \in X \mid f(p) \neq 0\}$

$$\operatorname{Div}(X) = \{f : X \to \mathbb{Z} \mid \operatorname{supp}(f) \text{ is finite}\}$$

We can write $D \in \operatorname{Div}(X)$ as a function $D : X \to \mathbb{Z}$, $p \mapsto D(p)$

or as a formal sum

$$D = \sum_{p \in X} n_p \cdot p \quad \text{where } n_p = D(p)$$

and all but finitely many n_p are zero.

[Identify divisor p with $\delta_p : X \to \mathbb{Z}$ $\quad \delta_p(q) = \begin{cases} 1 & q = p \\ 0 & q \neq p \end{cases}$]

Degree: The degree of a divisor is the sum of the coefficients

$$\deg(D) = \sum_{p \in X} D(p)$$

or $\deg(D) = \sum_{p \in X} n_p$ where $D = \sum_{p \in X} n_p \cdot p$

$\deg : \operatorname{Div}(X) \to \mathbb{Z}$ is a group homomorphism.

$\operatorname{Div}_0(X) := \ker(\deg)$ is group of divisors of degree 0.

div(f): A meromorphic function f has an associated divisor

$$\operatorname{div}(f) = \sum_{p \in X} \operatorname{ord}_p(f) \cdot p$$

Divisors of the form $\operatorname{div}(f)$ are called <u>principal divisors</u>: $\operatorname{PDiv}(X)$

Lemma $\left.\begin{array}{l} \mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g) \\ \mathrm{div}(1/f) = -\mathrm{div}(f) \\ \mathrm{div}(f/g) = \mathrm{div}(f) - \mathrm{div}(g) \end{array}\right\}$ follow from corresp. prop. of ord_p

Cor $\mathrm{div} : \mathcal{M}^* \longrightarrow \mathrm{Div}(X)$ is a group homomorphism

$\uparrow$ nonzero meromorphic functions

$$\mathrm{PDiv}(X) = \mathrm{image}(\mathrm{div}) \subset \mathrm{Div}(X) \text{ is a subgroup.}$$

Lemma $\deg(\mathrm{div}(f)) = 0$ hence $\mathrm{PDiv}(X) \subset \mathrm{Div}_0(X)$

Proof $\deg(\mathrm{div}(f)) = \sum_{p \in X} \mathrm{ord}_p(f) = \sum_{p \in X} \mathrm{Res}_p\left(\frac{df}{f}\right) = 0$ by residue theorem.

Example $X = \hat{\mathbb{C}}$ $f(z)$ rational function $f(z) = c \prod_{i=1}^{n} (z - \lambda_i)^{e_i}$

$$\mathrm{div}(f) = \sum_{i=1}^{n} e_i \lambda_i - \left(\sum_{i=1}^{n} e_i\right) \cdot \infty$$

Divisor of zeros: $\mathrm{div}_0(f) = \sum_{p,\, \mathrm{ord}_p(f) > 0} \mathrm{ord}_p(f) \cdot p$

Divisor of poles $\mathrm{div}_\infty(f) = \sum_{p,\, \mathrm{ord}_p(f) < 0} (-\mathrm{ord}_p(f)) \cdot p$

Thus $\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f)$

div(ω): if ω is a meromorphic 1-form, can do the same thing

$$\mathrm{div}(\omega) = \sum_{p \in X} \mathrm{ord}_p(\omega) \cdot p$$

Divisors of the form $\mathrm{div}(\omega)$ are called canonical divisors $\mathrm{KDiv}(X)$

Example: $X = \hat{\mathbb{C}}$ $\omega = dz$ on $\mathbb{C}$, in other chart $w = \frac{1}{z}$, $\omega = \frac{-1}{w^2}dw$
so $\operatorname{div}(\omega) = -2 \cdot \infty$

Lemma: f meromorphic function, ω meromorphic 1-form
$$\operatorname{div}(f\omega) = \operatorname{div}(f) + \operatorname{div}(\omega)$$

Lemma if ω_1 and ω_2 are two meromorphic 1-forms, then $\exists$ meromorphic function f such that $\omega_2 = f\omega_1$

Pf: write $\omega_i = f_i(z)dz$ in coords, and define $f = \frac{f_2}{f_1}$.

Thus $\operatorname{div}(\omega_2) - \operatorname{div}(\omega_1) = \operatorname{div}(f) \in \operatorname{PDiv}(X)$

Corollary: $\operatorname{KDiv}(X) \subset \operatorname{Div}(X)$ is a coset of the subgroup $\operatorname{PDiv}(X)$

$$\operatorname{KDiv}(X) = \operatorname{div}(\omega) + \operatorname{PDiv}(X) \quad \text{for any } \omega \neq 0$$

Degree of a canonical divisor: since $\operatorname{PDiv}(X) \subset \operatorname{Div}_0(X)$ all canonical divisors must have the same degree, but it depends on the genus.

Let f be a nonconstant meromorphic function on X, regard it as a map $F: X \to \hat{\mathbb{C}}$
let $\omega = dz$ be the 1-form on $\hat{\mathbb{C}}$ such that $\operatorname{div}(\omega) = -2 \cdot \infty$
then $\eta = F^*(\omega)$ is a meromorphic 1-form on X.

$$\deg(\operatorname{div}(\eta)) = \sum_{p \in X} \operatorname{ord}_p(F^*(\omega)) = \sum_{p \in X} \left[(1 + \operatorname{ord}_{F(p)}\omega) \operatorname{mult}_p(F) - 1 \right]$$

$$= \sum_{p \in X} (\operatorname{mult}_p(F) - 1) + \sum_{p \in F^{-1}(\infty)} -2 \cdot \operatorname{mult}_p(F)$$

$$= 2g - 2 + 2\deg(F) - 2\deg(F) = 2g - 2$$

where we have used Hurwitz' formula.
Thus, every canonical divisor has degree $2g-2$: $\deg(\mathrm{div}(\omega))=2g-2$

Inverse images: $F: X \to Y$ $\quad q \in Y$

$$F^*(q) = \sum_{p \in F^{-1}(q)} \mathrm{mult}_p(F) \cdot p$$

For $D = \sum_{q \in Y} n_q q \in \mathrm{Div}(Y)$, set $F^*(D) = \sum_{q \in Y} n_q F^*(q)$

Ramification divisor. $F: X \to Y$ $\quad R_F := \sum_{p \in X} [\mathrm{mult}_p(F) - 1] \cdot p \in \mathrm{Div}(X)$

Hurwitz' formula in divisor form: let ω be nonzero mero 1-form on Y

$$\mathrm{div}(F^*\omega) = F^*(\mathrm{div}(\omega)) + R_F$$

Projective curves: divisor of a homogeneous polynomial:
suppose $X \subset \mathbb{P}^n$ is holomorphically embedded.

$G(x_0, \ldots, x_n)$ homog poly of degree d that is not identically 0 on X.
At any $p \in X$, we can define $\mathrm{ord}_p(G)$ as follows: let H be a homog poly of degree d that does not vanish at p, and set

$$\mathrm{ord}_p(G) = \mathrm{ord}_p\left(\frac{G}{H}\right).$$ this doesn't depend on choice of H.
and $\mathrm{ord}_p(G) \geq 0$ always

define $\mathrm{div}(G) = \sum_{p \in X} \mathrm{ord}_p(G) \cdot p$ $\quad$ this is not a principal divisor because G is not actually a function.

$\mathrm{div}(G)$ is called an <u>intersection divisor</u> because it records the intersections of X with the projective hypersurface $\{G=0\} \subset \mathbb{P}^n$

<u>Properties:</u> $\mathrm{div}(G_1G_2) = \mathrm{div}(G_1) + \mathrm{div}(G_2)$

if $f = \frac{G}{H}$ is a meromorphic function on X,

$$\mathrm{div}(f) = \mathrm{div}(G) - \mathrm{div}(H)$$

When $G = a_0X_0 + \cdots + a_nX_n$ is a linear homog poly, $\mathrm{div}(G)$ is called a <u>hyperplane divisor</u>.

the difference of two hyperplane divisors is a principal divisor.

<u>Partial ordering:</u> Think of divisors as functions $D: X \to \mathbb{Z}$

$D \geq 0$ means $D(p) \geq 0$ for all $p \in X$

$D > 0$ means $D \geq 0$ and $D \neq 0$ so $(2p) > 0$

$D_1 \geq D_2$ means $D_1 - D_2 \geq 0$, similar for $>$, $\leq$, $<$.

thus $\mathrm{Div}(X)$ is partially ordered.

Observe, every D can be written uniquely as $D = P - N$ where $P, N > 0$ and P and N have disjoint support.

$$\min\{D_1, D_2, \ldots, D_r\}(p) := \min\{D_1(p), \ldots, D_r(p)\}$$

If f and g are nonzero meromorphic functions such that $f+g \neq 0$ then

$$\mathrm{div}(f+g) \geq \min\{\mathrm{div}(f), \mathrm{div}(g)\}$$