Geometric and topological structure of a Riemann surface.

Let us start with the following observations

(1) A holomorphic function $f: U \to V$ between open sets in $\mathbb{C}$ is always C^∞ as a map from two real variables to two real variables.

(2) If $f: U \to V$ is a holomorphic function and $f(z_0) \neq 0$, then

(a) f preserves angles at z_0

(b) f preserves orientation at z_0

Pf (1): Near $z_0 \in U$, $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

$\left(|z - z_0| < R = \text{radius of conv}\right)$ $\quad = \sum_{n=0}^{\infty} a_n (x + iy - z_0)^n$

is an absolutely convergent power series representation.

$\Rightarrow$ partial derivatives are given by term-by-term differentiation.

Pf (2): The matrix of partial derivatives of $f = u(x,y) + i v(x,y)$ has the form

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{by CR eqns}$$

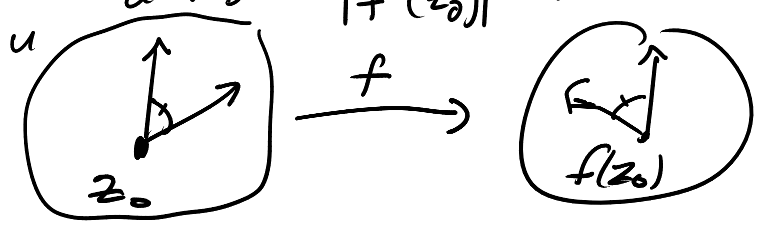
$$= \sqrt{a^2+b^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{where } \tan\theta = \frac{b}{a}$$

$$\left(\sqrt{a^2+b^2} = |f'(z_0)|\right)$$

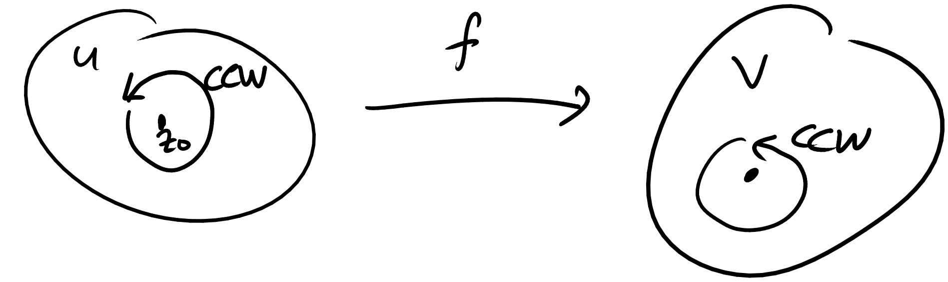
This matrix is a rotation composed w/ a scaling

Determinant $= a^2 + b^2 = |f'(z_0)|^2 > 0$

Preserves angles



and orientation



It follows from (1) that a complex structure on X determines a structure of a 2d C^∞-manifold on X. We call this the underlying smooth surface of the Riemann surface.

It follows from (2) that angles and orientation make sense at any point of a Riemann surface. Just measure in any chart — all charts give same answer.

Remarks: · We cannot intrinsically measure length on a R.S.

· Many different Riemann surfaces have same underlying smooth surface.

· FACT: The complex structure is determined by the local notion of angles and orientation

In particular, the underlying smooth surface of X comes with a preferred orientation. When X is compact, we have a classification of such smooth surfaces

Thm let X be a compact Riemann surface. Then there is a $g \geq 0$ such that the underlying smooth surface of X is a "torus with g holes". g is called the genus of X.

Examples of Riemann surfaces.

See the discussion on pp. 7-8 of the text. Basically this says that there is a unique topology that is compatible with a complex structure, so to define a Riemann surface, it suffices to give a set and a collection of complex charts. However, in doing this, one must check the Hausdorff property at the end.

Projective line Let $\mathbb{CP}^1$ be the set of 1-dimensional $\mathbb{C}$-vector subspaces of $\mathbb{C}^2$.

$$\mathbb{CP}^1 = \{ V \subset \mathbb{C}^2 \mid V \text{ is 1 dimensional subspace} \}$$

Let (z,w) be coordinates on $\mathbb{C}^2$

If $(z,w) \in \mathbb{C}^2$ is a nonzero vector, its $\mathbb{C}$-span is an element of $\mathbb{CP}^1$. We denote it by $[z:w] \in \mathbb{CP}^1$

Note that $\quad [z:w] = [\lambda z : \lambda w] \quad$ for $0 \neq \lambda \in \mathbb{C}$

So the map $\mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{CP}^1$ is many-to-one.

We can now give $\mathbb{CP}^1$ the quotient topology as a quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$

Atlas: Let $U_1 = \{ [z:w] \mid z \neq 0 \}$ $\qquad U_2 = \{ [z:w] \mid w \neq 0 \}$

- These sets cover $\mathbb{CP}^1$.
- Define $\varphi_0 : U_0 \to \mathbb{C}$ $\qquad \varphi_0([z:w]) = \dfrac{w}{z}$ (yes its well defined)

  $\varphi_1 : U_1 \to \mathbb{C}$ $\qquad \varphi_1([z:w]) = \dfrac{z}{w}$

$$\varphi_0(U_0 \cap U_1) = \mathbb{C}^* = \varphi_1(U_0 \cap U_1)$$

$\varphi_1 \circ \varphi_0^{-1} : \mathbb{C}^* \to \mathbb{C}^*$ is $\varphi_1 \circ \varphi_0^{-1}(x) = \frac{1}{x}$ which is holomorphic.

so φ_1 and φ_0 are compatible.

[Similar to $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In fact $\mathbb{CP}^1$ and $\hat{\mathbb{C}}$ are isomorphic]

<u>Complex Tori:</u> let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$

let $L = \{ m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z} \}$

be the lattice they span.

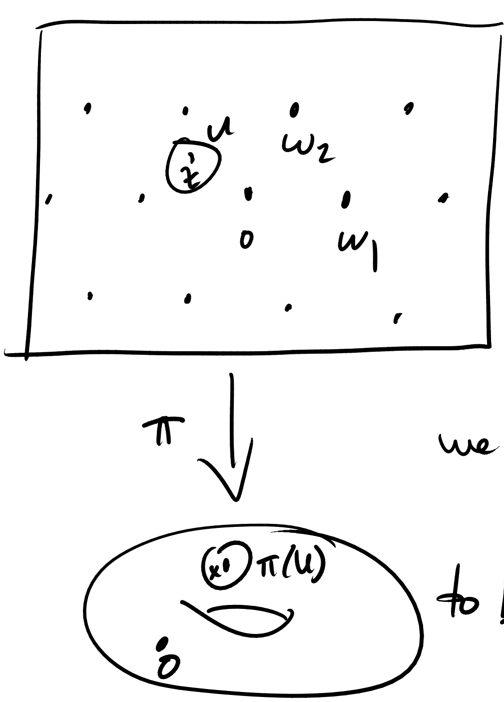
Let $X = \mathbb{C}/L$ be the quotient group (addition is group operation)

$\pi : \mathbb{C} \to \mathbb{C}/L$ quotient map.

Equip X with the quotient topology.

We can also equip X with a "quotient Riemann surface structure"

In this case:



Near $x \in \mathbb{C}/L$, choose z such that $\pi(z) = x$ and an open disk $U \ni z$ such that $\pi|_U : U \to \pi(U)$ is a bijection. Then we declare

$$(\pi|_U)^{-1} : \pi(U) \to U \subset \mathbb{C}$$

to be a chart.