Integration II:

Integration of 2-forms: The basic shape to integrate over is a triangle. Let $T \subset X$ be a triangle, that is, the image of a smooth map embedding $\tau: \Delta \to X$, where $\Delta = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\}$

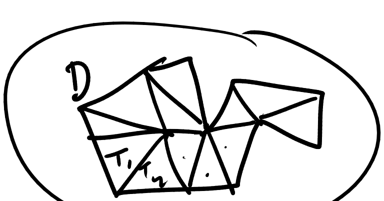
If T is contained entirely in the domain of a chart $\phi: U \to V$, and η is a smooth 2-form, we may define

$$\iint_T \eta = \iint_{\phi(T)} f(z,\bar{z})\, dz \wedge d\bar{z} = \iint_{\phi(T)} (-2i)\, f(x+iy, x-iy)\, dx \wedge dy$$

where $\eta = f(z,\bar{z})\, dz \wedge d\bar{z}$ is the local representation of η in the chart ϕ, and the right hand side is an ordinary double integral in $\mathbb{C} \simeq \mathbb{R}^2$.

If T is contained in the domain of two charts, then the integral has the same value w/r/t each chart precisely because of the transformation rule for 2-forms.

Suppose $D \subseteq X$ is a closed subset that can be written as a union of triangles in the above sense. If we choose a triangulation of D so fine that each triangle is contained in the domain of a chart, then, we can define



$$\iint_D \eta = \sum_i \iint_{T_i} \eta$$

The result does not depend on the choice of triangulation.

Chains & boundaries

A <u>0-simplex</u> in X is a map $\{*\} \to X$
(same thing as a point in X)

A <u>1-simplex</u> in X is a map $[0,1] \to X$
(same thing as a path)

A <u>2-simplex</u> in X is a map $\Delta \longrightarrow X$

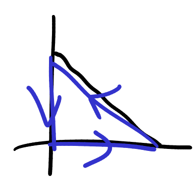
$$\Delta = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\}$$

(same thing as a triangle in X, w/ parametrization)

A <u>k-chain</u> in X is a formal sum of k-simplices $(k=0,1,2)$
The set of k-chains $C_k(X)$ is a $\mathbb{Z}$-module.

There are boundary operators $\partial : C_1(X) \to C_0(X)$ which are linear
$\partial : C_2(X) \to C_1(X)$
That associate to a k-simplex a $(k-1)$-chain which is its boundary.

$$\left(\gamma : [0,1] \to X, \text{ a 1-simplex}\right) \xrightarrow{\partial} \left(\gamma(1) - \gamma(0), \text{ a 0-chain}\right)$$

$\left(\tau : \Delta \to X, \text{ a 2-simplex}\right) \longrightarrow$ ( the sum of the boundary edges w/ ccw orientation, a 1-chain)

A k-chain $c \in C_k(X)$ such that $\partial c = 0$ is a <u>k-cycle</u> $Z_k(X)$
If $c = \partial d$ for some $(k+1)$-chain d, c is called k-boundary. $B_k(X)$
$\partial\partial = 0 \Rightarrow B_k(X) \subset Z_k(X)$, so we define $H_k(X) = Z_k(X)/B_k(X)$
called the Homology group of X.

Stokes's theorem:

<u>Lemma</u>: let $\tau: \Delta \to X$ be a smooth 2-simplex in X, and let ω be a smooth 1-form on X. Then

$$\int_{\partial\tau} \omega = \iint_{\tau} d\omega$$

<u>Proof</u>: Pulling ω and $d\omega$ back by τ, we have

$$\int_{\partial\tau} \omega = \int_{\partial\Delta} \tau^*\omega \quad , \quad \iint_{\tau} d\omega = \iint_{\Delta} \tau^* d\omega = \iint_{\Delta} d(\tau^*\omega)$$

writing $\omega = a(x,y)\,dx + b(x,y)\,dy$

$$d\omega = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy$$

So we are saying

$$\int_{\partial\Delta} a\,dx + b\,dy = \iint_{\Delta} \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy$$

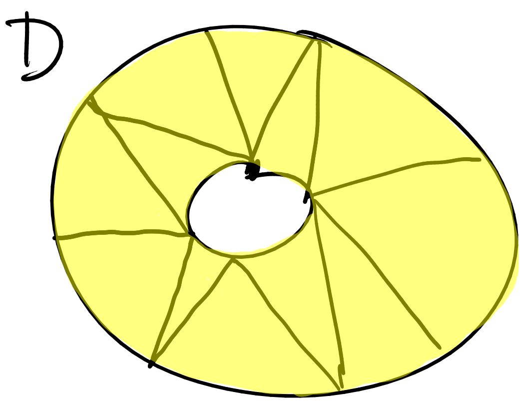
which is just Green's theorem in the plane $\blacksquare$

Now suppose $D \subset X$ is a triangulable subset. A triangulation of D gives us a 2-chain $\tilde{D} \in C_2(X)$ such that

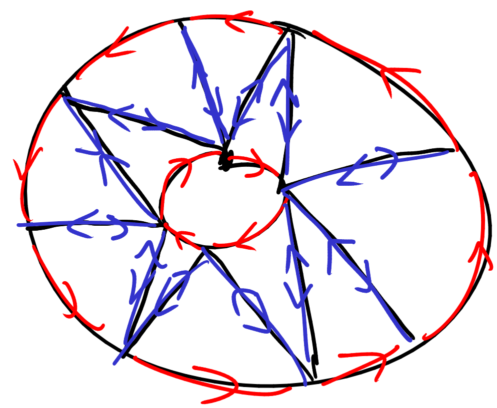
$$\iint_{D} \eta = \iint_{\tilde{D}} \eta \quad \text{for any 2-form } \eta.$$

Then by the above $\displaystyle \iint_{D} d\omega = \iint_{\tilde{D}} d\omega = \int_{\partial\tilde{D}} \omega$

Now the 1-chain $\partial\tilde{D}$ may have many terms corresponding to "inner edges" of the triangulation of D (those edges that touch two triangles) however, each such inner edge is counted twice, with opposite orientations (and possibly different parametrizations). In the integral, those terms cancel in pairs, and we are left with the integral over the "outer edges" of the triangulation of D



Triangulation of D



Blue : inner edges
Red : outer edges.

<u>Stokes's theorem</u>: If $D \subset X$ is a triangulable subset, ω a 1-form,

$$\int_{\partial D} \omega = \iint_D d\omega$$

where ∂D denotes a 1-chain formed by the "outer edges" of a triangulation of D.

<u>Corollary</u>: if $d\omega = 0$, then $\int_{\partial D} \omega = 0$ for any triangulable region D.

In particular, if ω is a holomorphic 1-form in a neighborhood of D, then $d\omega = 0$ in that neighborhood, so $\int_{\partial D} \omega = 0$

Now we come to a central result in the theory of compact R.S.

<u>Residue theorem</u>: Let ω be a meromorphic 1-form on a <u>compact</u> Riemann surface X. Then

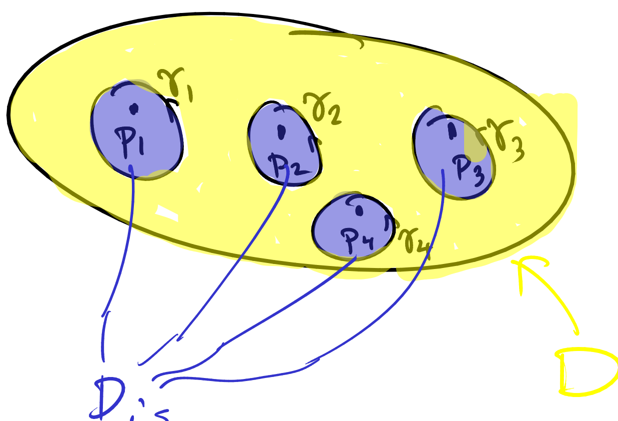
$$\sum_{p \in X} \operatorname{Res}_p(\omega) = 0$$

<u>Proof</u> $\operatorname{Res}_p(\omega) = \frac{1}{2\pi i} \int_\gamma \omega$, where γ is a small loop around p.

and it is only nonzero if p is a pole of ω. So:

$$\sum_{p \in X} 2\pi i \operatorname{Res}_p(\omega) = \sum_{\substack{\text{poles} \\ p_1, \dots, p_n}} \int_{\gamma_i} \omega \,, \quad \gamma_i \text{ a small loop around } p_i$$

The loops γ_i divide X into a union of disks D_i around p_i and another large region D



Also note that ω is holomorphic on D

And $\partial D = -\sum_{i=1}^{n} \gamma_i$

so $$\sum_{i=1}^{n} 2\pi i \operatorname{Res}_{p_i}(\omega) = \sum_{i=1}^{n} \int_{\gamma_i} \omega = \int_{\Sigma \gamma_i} \omega = \int_{-\partial D} \omega = -\int_{\partial D} \omega = -\iint_D d\omega = 0$$

Some further remarks

Two 1-chains c_1, c_2 are called homologous if $c_1 - c_2 \in B_1(X)$ is a boundary. Stokes's theorem implies that if ω is closed, $d\omega = 0$, then the integrals of ω over homologous chains have the same value.

If $d\omega = 0$, integration of ω defines a linear map $\int \omega : \dfrac{C_1(X)}{B_1(X)} \to \mathbb{C}$

usually we restrict this to cycles, giving a map

$$\int \omega : \frac{Z_1(X)}{B_1(X)} = H_1(X) \longrightarrow \mathbb{C}$$

If two paths are homotopic, they are also homologous, so we can also use a closed 1-form to get a map

$$\int \omega : \pi_1(X, p) \longrightarrow \mathbb{C}$$

<u>Fact</u>: If X is a connected compact Riemann surface,

$H_0(X) \cong \mathbb{Z}$

$H_1(X) \cong \mathbb{Z}^{2g}$ $\quad g =$ genus of X

$H_2(X) \cong \mathbb{Z}$