Contour integration on a Riemann surface.

Def A *path* on a R.S. X is a map $\gamma : [a,b] \to X$ that is continuous and piecewise smooth.

$\gamma(a)$: initial point

$\gamma(b)$: final point

γ is closed if $\gamma(a) = \gamma(b)$.

Reparametrization: If $\alpha : [c,d] \to [a,b]$ is a continuous piecewise smooth map with $\alpha(c) = a$, $\alpha(d) = b$ then $\gamma \circ \alpha : [c,d] \to X$ is a reparametrization of γ

(Note that α is not required to be injective)

Reversal: Given $\gamma : [a,b] \to X$ define $-\gamma : [a,b] \to X$

$$-\gamma(t) = \gamma(a+b-t)$$

Direct image: If $F : X \to Y$ is smooth, for instance holomorphic, then given $\gamma : [a,b] \to X$ a path, we obtain

$$F_*\gamma = F \circ \gamma : [a,b] \to Y \text{ a path in } Y.$$

Concatenation: suppose γ_1 and γ_2 are paths on X, and the final point of γ_1 = initial point of γ_2

then, reparametrize γ_1 so that its domain is $[0, \tfrac{1}{2}]$

reparametrize γ_2 so that its domain is $[\tfrac{1}{2}, 1]$

Let $\gamma : [0,1] \to X$ be the path such that $\gamma\big|_{[0,\frac{1}{2}]}$ is γ_1, and $\gamma\big|_{[\frac{1}{2},1]}$ is γ_2.

<u>Partition</u>: is the reverse of concatenation: $\gamma : [a,b] \longrightarrow X$
choose $a = a_0 < a_1 < \cdots < a_n = b$. Then let $\gamma_i = \gamma|_{[a_{i-1}, a_i]}$

We call $\{\gamma_i\}_{i=1}^{n}$ a <u>partition</u> of γ, and γ is the concatenation of $\{\gamma_i\}$.

<u>Lemma</u>: Let γ be a path on a Riemann surface X.
Then there is a partition of γ into $\{\gamma_i\}$ such that each γ_i has image in a particular chart domain on X.

<u>Proof</u>: Let $\{U_\alpha\}_{\alpha\in A}$ be the collection of all chart domains
then $\{\gamma^{-1}(U_\alpha)\}_{\alpha\in A}$ is a covering of $[a,b]$. Since $[a,b]$

is compact, finitely many $\gamma^{-1}(U_\alpha)$ suffice to cover it.
Each $\gamma^{-1}(U_\alpha)$ is a disjoint union of intervals,
so in fact $[a,b]$ is covered by finitely many intervals,
such that γ take each interval into a chart domain. ▨

<u>Integration</u>: Let ω be a smooth 1-form on X. Let $\gamma : [a,b] \to X$
be a path. Choose partition $\{\gamma_i : [a_{i-1}, a_i] \to X\}$
so that γ_i has image in a chart domain U_i.
Let $\phi_i : U_i \to \mathbb{C}$ be corresponding chart.
Let z_i be the coordinate defined by ϕ_i in U_i

write $\phi_i \circ \gamma_i (t) = z_i(t)$

write $\omega = f_i(z_i, \overline{z_i})\, dz_i + g_i(z_i, \overline{z_i})\, d\overline{z_i}$ in U_i

Define $\int_\gamma \omega = \sum_{i=1}^{n} \int_{t=a_{i-1}}^{t=a_i} [f_i(z_i(t), \overline{z_i(t)}) z_i'(t) + g_i(z_i(t), \overline{z_i(t)}) \overline{z_i'(t)}]\, dt$

$$= \sum_{i=1}^{n} \int_{\phi_i \circ \gamma_i} f\, dz_i + g\, d\bar{z}_i$$

The result is independent of the local coordinates and partition used to define it.

Basic properties: (a) If $\alpha : [c,d] \to [a,b]$ is a reparametrization

$$\int_{\gamma \circ \alpha} \omega = \int_\gamma \omega$$

(b) $\mathbb{C}$-linearity:

$$\int_\gamma (\lambda \omega_1 + \mu \omega_2) = \lambda \int_\gamma \omega_1 + \mu \int_\gamma \omega_2$$

(c) FTC: If f is a C^∞ function defined in a neighborhood of the image of $\gamma : [a,b] \to X$, then

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

(d) Linear w/r/t partition of path: If $\{\gamma_i\}$ is a partition of γ,

$$\int_\gamma \omega = \sum_{i=1}^{n} \int_{\gamma_i} \omega$$

(e) If $-\gamma$ denotes the reversal of γ,

$$\int_{-\gamma} \omega = - \int_\gamma \omega$$

(f) Adjunction. If $F: X \to Y$ is a smooth map, then F_* on paths is adjoint to F^* on forms.

$$\int_{F_*\gamma} \omega = \int_\gamma F^*\omega$$

<u>Chains</u>: A 1-chain on X is a formal sum of paths with integer coefficients

$$C = \sum_{i=1}^{k} n_i \gamma_i \qquad \gamma_i \text{ paths.}$$

$CH_1(X)$ = free abelian group generated by all paths.

We can extend the notion of integration from paths to 1-chains

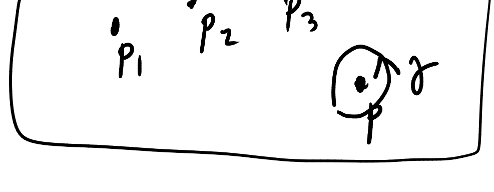
$$\int_C \omega := \sum_{i=1}^{k} n_i \int_{\gamma_i} \omega.$$

There are also 0-chains, which are formal $\mathbb{Z}$-linear combinations of points, and 2-chains, which are formal $\mathbb{Z}$-linear combinations of maps $\Delta \longrightarrow X$

$$\Delta = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} x \geq 0 \\ y \geq 0 \end{matrix} \quad x+y \leq 1 \right\}$$

This leads to the notion of Homology.

<u>Residues</u>: Let ω be a meromorphic 1-form on X. Let $p \in X$ be a pole
→ choose a small loop γ enclosing p and not enclosing any other poles of ω.



Definition: The residue of ω at p is $Res_p(\omega) = \frac{1}{2\pi i}\int_\gamma \omega$

Let z be a local coordinate near p. Write ω as

$$\omega = \left(\sum_{n=M}^{\infty} c_n z^n\right) dz.$$

Lemma $Res_p(\omega) = c_{-1}$. In particular, c_{-1} has the same value w/r/t different coordinates. and previous def doesn't depend on choice of small loop enclosing p.

Proof in local coordinate z, $\int_\gamma \omega = \int_{\phi\circ\gamma} f(z)dz = 2\pi i c_{-1}$

by the ordinary residue theorem in the complex plane.

Lemma: Suppose f is a meromorphic function defined near $p \in X$ then $\frac{df}{f}$ is a meromorphic 1-form and

$$Res_p\left(\frac{df}{f}\right) = ord_p(f)$$

Proof $f = \sum_{n=M}^{\infty} c_n z^n$ $df = \left(\sum_{n=M}^{\infty} n c_n z^{n-1}\right) dz$

Thus $\frac{1}{f} = c_M^{-1} z^{-M} + (\text{higher order terms})$

So $\frac{df}{f} = \left(M c_M z^{M-1} \cdot c_M^{-1} z^{-M} + h.o.t.\right) dz$

$= \left(M z^{-1} + h.o.t.\right) dz$

$Res_p\left(\frac{df}{f}\right) = M = ord_p(f)$. ∎