Operations on differential forms.
We define the operations in local coordinate representations.

* $h(z,\bar{z})$ function, $\omega = f(z,\bar{z})dz + g(z,\bar{z})d\bar{z}$ 1-form

$$h\cdot\omega = h\cdot f\cdot dz + h\cdot g\cdot d\bar{z} \quad \text{is a 1-form.}$$

Observe
- ω type $(1,0) \Rightarrow h\omega$ type $(1,0)$
- ω type $(0,1) \Rightarrow h\omega$ type $(0,1)$
- ω holomorphic & h holomorphic $\Rightarrow h\omega$ holomorphic
- same for meromorphic.
- ω, h meromorphic $\Rightarrow \mathrm{ord}_p(h\omega) = \mathrm{ord}_p(h) + \mathrm{ord}_p(\omega)$.

* Also $h(z,\bar{z})$ function, $\eta = f(z,\bar{z})dz\wedge d\bar{z}$ 2-form
$h\eta = h\cdot f\cdot dz\wedge d\bar{z}$ 2-form.

* Wedge product: Rule: $dz\wedge d\bar{z} = -d\bar{z}\wedge dz$
$dz\wedge dz = 0 = d\bar{z}\wedge d\bar{z}$

Thus if $\omega_1 = f_1(z,\bar{z})dz + g_1(z,\bar{z})d\bar{z}$
$\omega_2 = f_2(z,\bar{z})dz + g_2(z,\bar{z})d\bar{z}$

$$\begin{aligned}
\omega_1\wedge\omega_2 &= (f_1\,dz + g_1\,d\bar{z})\wedge(f_2\,dz + g_2\,d\bar{z}) \\
&= f_1f_2\,dz\wedge dz + f_1g_2\,dz\wedge d\bar{z} + g_1f_2\,d\bar{z}\wedge dz + g_1g_2\,d\bar{z}\wedge d\bar{z} \\
&= 0 + f_1g_2\,dz\wedge d\bar{z} - g_1f_2\,dz\wedge d\bar{z} + 0 \\
&= (f_1g_2 - g_1f_2)\,dz\wedge d\bar{z}.
\end{aligned}$$

This is a well-defined 2-form. (Need to check something)

Differential operators on forms.

*: operators : functions $\to$ forms.

Recall $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

Given a smooth function $f(z,\bar{z})$, define

$$\partial f = \frac{\partial f}{\partial z} dz \qquad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

$$df = \partial f + \bar{\partial} f = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

<u>Lemma</u> the above defines 1-forms $\partial f, \bar{\partial} f, df$.

f is holomorphic iff $\bar{\partial} f = 0$.

The operators $\partial, \bar{\partial}, d$ are $\mathbb{C}$-linear and satisfy the product rule

$$d(fg) = f\,dg + g\,df, \quad \partial(fg) = f\partial g + g\partial f, \quad \bar{\partial}(fg) = f\bar{\partial} g + g\bar{\partial} f.$$

* operators 1-forms $\to$ 2 forms.

Given $\omega = f(z,\bar{z})\,dz + g(z,\bar{z})\,d\bar{z}$

Define $\partial\omega = (\partial f)\wedge dz + (\partial g)\wedge d\bar{z} = \frac{\partial f}{\partial z} dz \wedge dz + \frac{\partial g}{\partial z} dz \wedge d\bar{z}$

or just $\partial \omega = \frac{\partial g}{\partial z} dz \wedge d\bar{z}$

$$\bar{\partial}\omega = (\bar{\partial} f)\wedge dz + (\bar{\partial} g)\wedge d\bar{z} = \frac{\partial f}{\partial \bar{z}} d\bar{z}\wedge dz + \frac{\partial g}{\partial \bar{z}} d\bar{z} \wedge d\bar{z}$$

or just $\bar{\partial}\omega = -\frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}$

$$d\omega = \partial\omega + \bar{\partial}\omega = \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right) dz \wedge d\bar{z}$$

Lemma: these local expressions define 2-forms $d\omega, \partial\omega, \bar\partial\omega$

A 1-form ω is holomorphic iff it has type $(1,0)$ and $\bar\partial\omega = 0$.
The operators are $\mathbb{C}$-linear and satisfy, where f is a function and ω is a form:

$$d(f\omega) = df \wedge \omega + f\,d\omega \qquad \text{And} \quad ddf = 0$$
$$\partial(f\omega) = \partial f \wedge \omega + f\,\partial\omega \qquad \partial\partial f = 0$$
$$\bar\partial(f\omega) = \bar\partial f \wedge \omega + f\,\bar\partial\omega. \qquad \bar\partial\bar\partial f = 0.$$

Also $\partial\bar\partial f = -\bar\partial\partial f$.

Def: ω is d-closed if $d\omega = 0$, and d-exact if $\omega = df$ for some f.
Define ∂-closed/exact, $\bar\partial$-closed/exact similarly.

Since $ddf = 0$, exact $\Rightarrow$ closed, but not vice versa.

Poincaré Lemma: Let ω be a smooth 1-form in a neighborhood U of p. Suppose $d\omega = 0$ in U (ω is closed) then in some possibly smaller neighborhood V of p, there is a function f such that $\omega = df$ in V. (ω is exact in V)

Dolbeault Lemma: Let ω be a smooth form of type $(0,1)$. Then there is a neighborhood V of p and a function f in V such that $\bar\partial f = \omega$.

Remark: Poincaré Lemma involves finding a function with prescribed derivatives, so it's basically the fundamental theorem of calculus. Dolbeault's Lemma is deeper, since it involves solving a partial differential equation. It is roughly at the level of solving Poisson's equation $\Delta u = f$.

Pullbacks: let $F : X \to Y$ be a holomorphic map.
If $f : Y \to \mathbb{C}$ is a function, the pullback $F^*(f) : X \to \mathbb{C}$ is simply $F^*(f) = f \circ F$

To pullback a 1-form, choose a chart $\phi : U \to V$ on X so that $F(U)$ is contained in the domain of a chart $\psi : U' \to V'$ on Y. Then we have coordinates z on $U' \subset Y$ and w on $U \subset X$.
The map F is represented locally as a holomorphic function

$$z = h(w)$$

The basic rule is $dz = h'(w)\,dw$
or more precisely $F^*(dz) = h'(w)\,dw$
Generally, if $\omega = f(z,\bar z)\,dz + g(z,\bar z)\,d\bar z$

$$F^*(\omega) = f(h(w), \overline{h(w)})\, h'(w)\,dw + g(h(w), \overline{h(w)})\, \overline{h'(w)}\, d\bar w$$

Observations:
- ω holomorphic $\Longrightarrow F^*(\omega)$ is holomorphic
- ω meromorphic $\Longrightarrow F^*(\omega)$ is meromorphic
- ω of type $(1,0) \Rightarrow F^*(\omega)$ of type $(1,0)$
- ω of type $(0,1) \Rightarrow F^*(\omega)$ of type $(0,1)$

Pullback of 2-forms: if $\eta = f(z,\bar z)\,dz \wedge d\bar z$

$$F^*(\eta) = f(h(w), \overline{h(w)})\, \underbrace{h'(z)\,\overline{h'(z)}}_{|h'(z)|^2}\, dw \wedge d\bar w$$

Naturality of derivatives

$$F^*(df) = dF^*(f) \qquad F^*(d\omega) = dF^*(\omega)$$
$$F^*(\partial f) = \partial F^*(f) \qquad F^*(\partial \omega) = \partial F^*(\omega)$$
$$F^*(\bar\partial f) = \bar\partial F^*(f) \qquad F^*(\bar\partial \omega) = \bar\partial F^*(\omega)$$

Recall : $f: Y \to \mathbb{C}$ function $\quad \mathrm{ord}_p(F^*f) = \mathrm{ord}_{F(p)}(f) \cdot \mathrm{mult}_p(F)$

For 1-forms, this formula looks like:

$$\mathrm{ord}_p(F^*\omega) = \mathrm{ord}_{F(p)}(\omega) \cdot \mathrm{mult}_p(F) + \mathrm{mult}_p(F) - 1$$

Proof: Choose coordinates w centered at $p \in X$ and z centered at $F(p) \in Y$ such that F has the local representation $z = w^n$ $\quad n = \mathrm{mult}_p(F)$

Suppose $k = \mathrm{ord}_{F(p)}(\omega)$ so $\omega = (az^k + \text{h.o.t.})\,dz$

Then $F^*\omega = (a(w^n)^k + \text{h.o.t.})\, n w^{n-1} dw$

so $\mathrm{ord}_p(F^*\omega) = nk + n - 1$, as desired.