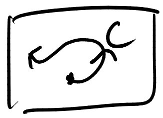
Differentials and integration.

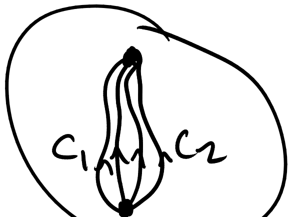
Recall that in complex analysis one considers contour integrals

$\int_C f(z)dz$ where $C \subset \mathbb{C}$ is a contour, what we would now call a path. If $\gamma : [0,1] \to \mathbb{C}$ is a parametrization of C, we set



$$\int_C f(z)dz = \int_0^1 f(\gamma(t))\, \gamma'(t)\, dt$$

The result does not depend on the parametrization, and If $f(z)$ is a holomorphic function, and two contours C_1, C_2 are homotopic rel endpoints, inside the domain of holomorphicity of f, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$



We now want to generalize this to Riemann surfaces.

Q: what sort of thing is the integrand $f(z)dz$? Suppose $w = T(z)$ is a holomorphic change of coordinates. let $\gamma : [0,1] \to \mathbb{C}$ be a contour (in the z-coordinate) Then $\tilde{\gamma} = T \circ \gamma$ is the representation of this contour in the w-coordinate, and $g(w) = f \circ T^{-1}(w)$ is the representation of f in the w coordinate.

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\, \gamma'(t)\, dt$$

$$\int_{\tilde{\gamma}} g(w)dw = \int_0^1 g(\tilde{\gamma}(t))\, \tilde{\gamma}'(t)\, dt$$

these are <u>not</u> equal.

$g(\tilde{\gamma}(t)) = (f \circ T^{-1})(T \circ \gamma(t)) = f(\gamma(t))$ so that matches,

but $\tilde{\gamma}'(t) = \frac{d}{dt} T(\gamma(t)) = \frac{dT}{dz}(\gamma(t))\gamma'(t)$

So there is an extra factor of $\frac{dT}{dz}$!

But $\displaystyle\int_{\tilde{\gamma}} g(w)\, dw = \int_{\gamma} f(z) \frac{dT}{dz}(z)\, dz$ is true.

One can also notate $\frac{dw}{dz} = \frac{dT}{dz} = T'(z)$.

Thus the correct change of variables rule is

$$dw \to \frac{dw}{dz} dz = T'(z)dz$$

$$g(w)dw \to g(T(z))\,T'(z)\,dz.$$

This motivates the following definition:

<u>Def</u>: Let V_1 and V_2 be open sets in $\mathbb{C}$. Let $T: V_1 \to V_2$ be a holomorphic isomorphism. Denote by z the coordinate in V_1, w the coordinate in V_2. Let $f(z): V_1 \to \mathbb{C}$ $g(w): V_2 \to \mathbb{C}$ be holomorphic functions. We say that <u>f transforms to g as a holomorphic 1-form (or differential) under T</u> if

$$f(z) = g(T(z))\,T'(z)$$

<u>Def</u>: Let X be a Riemann surface. <u>A holomorphic 1-form (or differential)</u> is the assignment, for each chart $\phi_\alpha: U_\alpha \to V_\alpha$ on X, of a holomorphic function $f_\alpha: V_\alpha \to \mathbb{C}$, satisfying the condition:

Suppose $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ are charts such that $U_1 \cap U_2 \subset X$ is not empty, let $T = \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ be the transition map. Let f_1 and f_2 be the holomorphic functions associated to the charts ϕ_1 and ϕ_2. Then we require that $f_1|_{\phi_1(U_1 \cap U_2)}$ transforms to $f_2|_{\phi_2(U_1 \cap U_2)}$ under T,

$$f_1 = (f_2 \circ T) \cdot T'.$$

As a matter of notation, when speaking of 1-forms we usually write z_α for the coordinate chart $\phi_\alpha : U_\alpha \to V_\alpha$, and we write $f_\alpha(z_\alpha)\, dz_\alpha$ for the representation of the one form in the chart ϕ_α. Thus, we can describe a 1-form as a collection of expressions $\{ f_\alpha(z_\alpha)\, dz_\alpha \}$ one for each chart, subject to the requirement that $f_\alpha(z_\alpha)\, dz_\alpha = f_\beta(z_\beta)\left(\frac{dz_\alpha}{dz_\beta}\right) dz_\beta$

whenever the transition $z_\beta \to z_\alpha$ is defined.

Note that, to define a holomorphic one form on X, it suffices to define the local expression $f(z)dz$ for a collection of charts covering X so that the local expressions transform to each other under appropriate transition maps. One can then define the local expression in all other charts by the transformation rule.

<u>Meromorphic 1-forms</u>: The definition is the same as for holomorphic 1-forms, but with local expressions $f(z)dz$ where $f(z)$ is meromorphic, rather that holomorphic.

Let ω be a meromorphic one form. Given a local coordinate z, write $\omega = f(z)dz$ with $f(z)$ meromorphic function. For any point p in the domain of the coord z, define

$$\mathrm{ord}_p(\omega) := \mathrm{ord}_{z_0}(f) \quad \text{where } z_0 \text{ corresponds to } p.$$

Well-defined: let z and w be two local coords with $p \in X$ in the domain of both. Then $\omega = f(z)dz$ in z-chart
$\omega = g(w)dw$ in w-chart

Let $z = T(w)$ be the transition map then

$$f(z)dz = f(T(w))\,T'(w)\,dw = g(w)\,dw$$
$$g(w) = f(T(w))\,T'(w)$$

$$\mathrm{ord}_{w_0}(g(w)) = \mathrm{ord}_{w_0}(f(T(w))) + \mathrm{ord}_{w_0} T'(w)$$

$$= \mathrm{ord}_{z_0}(f) \cdot \mathrm{mult}_{w_0}(T) + \mathrm{ord}_{w_0} T'$$

Since T is a holomorphic isomorphism $\mathrm{mult}_{w_0}(T) = 1$ and $\mathrm{ord}_{w_0} T' = 0$
so $\mathrm{ord}_{w_0}(g(w)) = \mathrm{ord}_{z_0}(f(z))$.

ω has zero of order $n \iff \mathrm{ord}_p(\omega) = n > 0$
ω has pole of order $n \iff \mathrm{ord}_p(\omega) = -n < 0$.
Zeros and poles are discrete.

Defining a meromorphic 1-form: The Identity theorem holds for meromorphic forms; if ω_1 and ω_2 are equal on an open set U in the (connected) R.S. X, then they are equal on all of X. Thus to define a meromorphic form, it suffices to give its representation in any one coordinate system. Note we are not gaurenteed that the definition extends to the rest of X, but if it does, the extension is unique.

Eg $X = \hat{\mathbb{C}}$ coordinate charts z on $\hat{\mathbb{C}} \setminus \{\infty\}$, w on $\hat{\mathbb{C}} \setminus \{0\}$ $\quad z = \frac{1}{w}$

let $\omega = dz$ on $\hat{\mathbb{C}} \setminus \{\infty\}$ then in w-coord $dz = \frac{-1}{w^2} dw$

$\frac{dz}{dw} = -\frac{1}{w^2}$

Thus ω has no zeros, and a single pole of order 2 at $w=0$, which is $\infty \in \hat{\mathbb{C}}$.

let $\omega = e^z dz$ on $\hat{\mathbb{C}} \setminus \{\infty\}$ then in w-coord: $e^z dz = e^{1/w} \frac{-1}{w^2} dw$
thus ω has an essential singularity at ∞, is not meromorphic.

Other kinds of forms: Smooth 1-forms are local expressions

$$\omega = f(x,y)\,dx + g(x,y)\,dy$$

where x and y are real and imag parts $z = x+iy$.
where $f(x,y)$ and $g(x,y)$ are smooth (C^∞) functions.

Since $z = x+iy$, $\bar{z} = x - iy$, so $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

In terms of differentials $dz = dx + i\,dy$, $d\bar{z} = dx - i\,dy$

$$dx = \frac{dz + d\bar{z}}{2} \qquad dy = \frac{dz - d\bar{z}}{2i}$$

Thus $f(x,y)\,dx + g(x,y)\,dy$ can be rewritten as
$$r(z,\bar{z})\,dz + s(z,\bar{z})\,d\bar{z}$$

Partial derivatives: formally

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

So let $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ $\quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

Observe: $\frac{\partial f}{\partial \bar{z}} = 0 \iff f$ is holomorphic.

A smooth one forms transform according to the rule that if
$z = T(w)$, then $dz = T'(w)\,dw$, $d\bar{z} = \overline{T'(w)}\,d\bar{w}$

$f_1(z,\bar{z})\,dz + g_1(z,\bar{z})\,d\bar{z}$ transforms to $f_2(w,\bar{w})\,dw + g_2(w,\bar{w})\,d\bar{w}$ if
$$f_2(w,\bar{w}) = f_1(T(w), \overline{T(w)})\,T'(w)$$
$$g_2(w,\bar{w}) = g_1(T(w), \overline{T(w)})\,\overline{T'(w)}.$$

<u>Type of forms</u>: A smooth 1-form $f(z,\bar{z})\,dz$ has <u>type</u> $(1,0)$
while $g(z,\bar{z})\,d\bar{z}$ has type $(0,1)$
Holomorphic forms have type $(1,0)$.

2-forms: A smooth 2-form has local expression $f(x,y)\,dx\wedge dy$
(where $dx\wedge dy = -dy\wedge dx$. $dx\wedge dx = dy\wedge dy = 0$)

Now $dz\wedge d\bar{z} = (dx+idy)\wedge(dx-idy) = i\,dy\wedge dx - i\,dx\wedge dy$
$= -2i\,dx\wedge dy$

So we can write a smooth 2-form as $f(z,\bar{z})\,dz\wedge d\bar{z}$

This transforms under $z = T(w)$ by $dz = T'(w)dw$, $d\bar{z} = \overline{T'(w)}d\bar{w}$.

$$dz\wedge d\bar{z} = T'(w)\overline{T'(w)}\,dw\wedge d\bar{w} = |T'(w)|^2\,dw\wedge d\bar{w}$$

Thus $f(z,\bar{z})dz\wedge d\bar{z}$ transforms to $g(w,\bar{w})dw\wedge d\bar{w}$ if
$$g(w,\bar{w}) = f\left(T(w),\overline{T(w)}\right)|T'(w)|^2$$