Monodromy II:

let $F: X \to Y$ be a nonconst. map between compact R.S.
If F has ramification points, it is not a covering map.

let $R = \{p \in X \mid \text{mult}_p F > 1\} \subset X$ be the set of ramification points. let $B = F(R) \subset Y$
let $V = Y \setminus B$ and $U = X \setminus F^{-1}(B)$

[Note $F^{-1}(B) = F^{-1}(F(R)) \supseteq R$ but may not be equal to it]
Thus $F|_U : U \to V$ is an unramified hol map, U, V not compact
By local structure of holomorphic maps $F|_U$ is a covering map of degree $\deg(F) =: d$.

Now apply theory of monodromy: Pick basepoint $q \in V$
Get monodromy representation $\rho : \pi_1(V, q) \to S_d$
X connected $\Rightarrow U$ connected $\Rightarrow \operatorname{im}(\rho)$ transitive.

We call this the monodromy representation of the map F.

We now claim there is a bijection

$$\left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of connected coverings}\\ F: U \to V\\ \text{of degree } d\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{group homomorphisms}\\ \rho : \pi_1(V, q) \to S_d\\ \text{with transitive image}\\ (\text{up to conjugacy in } S_d)\end{array}\right\}$$

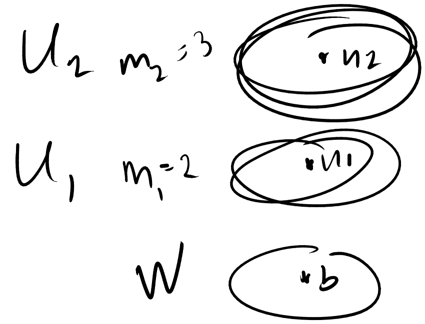
In one direction we take monodromy. In the other, if ρ is given, let $H := \ker(\rho) \subset \pi_1(V, q)$. Then let $U = U_0 / H$, where U_0 is the universal covering of V. The monodromy rep of this covering will be conjugate to ρ.

Another general fact is that if $F: U \to V$ is a covering, and V is a Riemann surface, there is a unique complex structure on U such that F is holomorphic. (if $W \subset U$ is an open set so small that $F|_W : W \to V$ is homeo onto its image, let ϕ be a chart on V, and let $\phi \circ F|_W$ be a chart on W.) So we conclude

<u>Prop</u>. For a Riemann surface V, there is a bijection

$$\left\{ \begin{array}{l} \text{iso classes of} \\ \text{unramified hol maps} \\ F: U \to V \\ \text{of degree } d \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{homomorphisms} \\ \rho: \pi_1(V, q) \to S_d \\ \text{with transitive image} \\ \text{up to conjugacy in } S_d \end{array} \right\}.$$

<u>What happens near branch points?</u> For $b \in B \subset Y$ a branch pt., let $W \ni b$ be a small open disk. Denote $u_1, \dots, u_k$ the points of $F^{-1}(b)$, and let $m_i = \text{mult}_{u_i}(F)$.
By local structure of holomorphic maps, we can choose W small enough that $F^{-1}(W)$ is the union of disks $U_1, \dots, U_k$ s.t. $u_i \in U_i$ and local coords z_i on U_i and w on W so that F has the form $w = z_i^{m_i}$ on U_i.



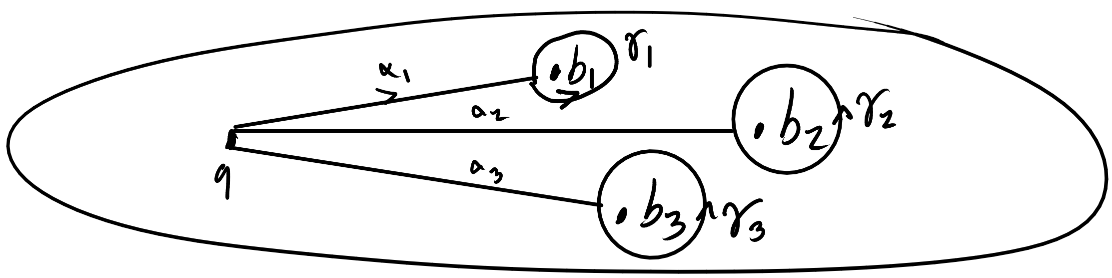
Let γ be a small loop in $W \setminus \{b\}$ that winds once around b. By monodromy construction we get a permutation σ of $F^{-1}(\gamma(0))$.

$$F^{-1}(\gamma(0)) = \coprod_{i=1}^{k} \left(U_i \cap F^{-1}(\gamma(0)) \right)$$

σ preserves each subset $U_i \cap F^{-1}(\gamma(0))$, and in fact acts on this set by cyclic permutation. We can deduce this from the local model $w = z_i^{m_i}$ on U_i

Thus, the permutation σ decomposes into disjoint cycles of lengths $m_1, m_2, \dots, m_k$.

If the base point q is not contained in W, its no problem, we can let α be a path from q to $\gamma(0) = \gamma(1) \in W$, and then consider the loop $\alpha^{-1}\gamma\alpha$ based at q.



Now we want to reconstruct the whole holomorphic map $F: X \to Y$ from its unramified part $F: U \to V$.
Near a branch point $b \in Y$, we have a small punctured disk $\tilde{W}$. It is the domain of hole chart on U. Suppose the cycle structure of the monodromy around b is $(m_1, \dots, m_k)$. Then, by the classification of covering spaces of the punctured disk, the preimage $F^{-1}(\tilde{W})$ is a disjoint union of connected coverings $\tilde{U}_i \to \tilde{W}$ of degree m_i. We know in this case that $\tilde{U}_i$ is also a punctured disk, so it defines a hole chart on U. Filling these holes, we can add k points mapping to b. Doing this for every branch point, we can recover X, and the map $F: X \to Y$.

This gives us the inverse to the map $\{F: X \to Y\} \to \{\rho: \pi_1(V, q) \to S_d\}$ and establishes the following correspondence:

Prop: let Y be a compact R.S., $B \subset Y$ a finite subset, let $q \in Y \setminus B$. there is a bijection

$$\left\{ \begin{array}{l} \text{iso. classes of} \\ \text{hol. maps } F: X \to Y \\ \text{of degree } d \text{ whose} \\ \text{brach set is contained in } B \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{group homomorphisms} \\ \rho: \pi_1(Y \setminus B, q) \to S_d \\ \text{with transitive image} \\ \text{up to conjugacy in } S_d \end{array} \right\}$$

(Note we say "contained" in B because the monodromy could be trivial around certain points $b \in B$, in which case the corresponding map is unramified at b.)

<u>Maps to $\mathbb{P}^1$</u>: If we let $Y = \mathbb{P}^1$, $B = \{b_1, \ldots, b_n\}$ then $\pi_1(\mathbb{P}^1 \setminus \{b_1, \ldots, b_n\}, q)$ is generated by n loops $\gamma_1, \ldots, \gamma_n$ around the points b_i, subject to the relation $[\gamma_1][\gamma_2] \cdots [\gamma_n] = 1$. (Thus it is isomorphic to a free group on $n-1$ generators.)

The monodromy data is there for a collection $\sigma_1, \ldots, \sigma_n \in S_d$ such that $\sigma_1 \cdots \sigma_n = 1$, and the subgroup generated by $\{\sigma_i\}_{i=1}^n$ is transitive.

Prop:
$$\left\{ \begin{array}{l} \text{iso. classes of hol. maps} \\ F: X \to \mathbb{P}^1 \text{ of degree } d \\ \text{with branch points contained in } B \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{conjugacy classes of } n\text{-tuples} \\ (\sigma_1, \ldots, \sigma_n) \text{ in } S_d \text{ such} \\ \text{that } \sigma_1 \cdots \sigma_n = 1 \text{ and} \\ \langle \sigma_1, \ldots, \sigma_n \rangle \subset S_d \text{ is} \\ \text{transitive} \end{array} \right\}$$

Furthermore, if the cycle structure of σ_i is $(m_{i1}, \ldots, m_{ik_i})$, then $F^{-1}(b_i)$ consists of points u_{ij} $(j = 1, \ldots, k_i)$ such that $\mathrm{mult}_{u_{ij}}(F) = m_{ij}$