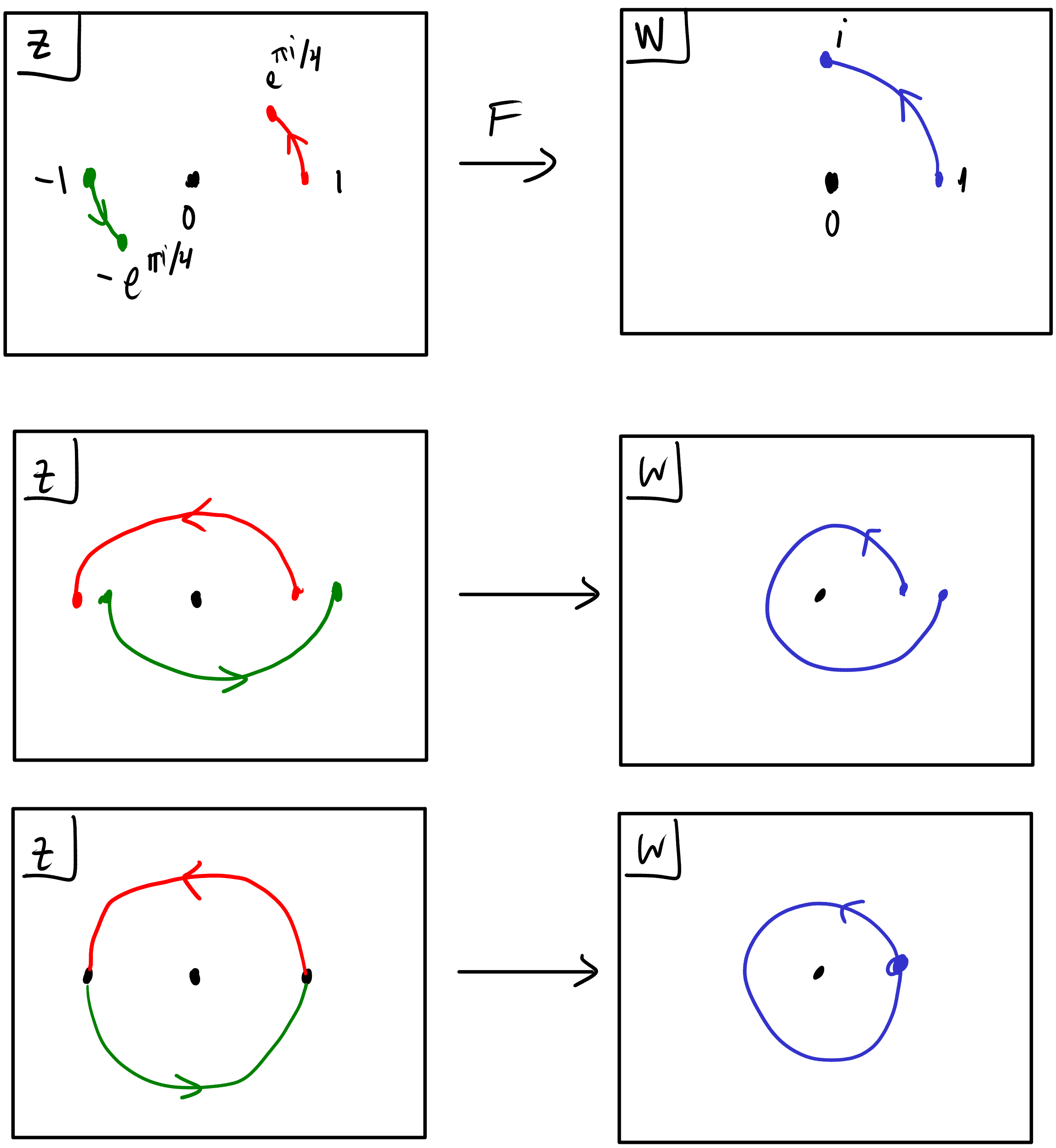
Monodromy: $F: X \to Y$ holomorphic map.
consider moving a point in Y, and try to "follow" its preimages. We can use this to understand X and F better.

Elementary example $F: \underset{\text{source}}{\mathbb{C}} \to \underset{\text{target}}{\mathbb{C}}$
$z \mapsto z^2 = w$

each w in the target $\mathbb{C}$ (except for 0), has two preimages
Consider how the preimages $F^{-1}(w)$ move when we move w?



preimages get swapped ← Go around loop

Covering spaces and Fundamental group. (Review)

V a "Reasonable" topological space, such as a real manifold

<u>path</u> in V : continuous map $\gamma : [0,1] \to V$

$\gamma(0) =$ start point, $\gamma(1) =$ endpoint.

<u>loop</u> in V based at q : path $\gamma : [0,1] \to V$ s.t. $\gamma(0) = \gamma(1) = q$.

loops γ_0 and γ_1 based at q are homotopic if:

$\exists$ continuous $G : [0,1] \times [0,1] \to V$

s.t. $G(0,t) = \gamma_0(t)$ $\quad G(1,t) = \gamma_1(t)$ $\quad \forall t$

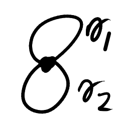
and $G(s,0) = G(s,1) = q$ $\quad \forall s$

Homotopy is an equivalence relation on loops based at q.

The fundamental group based at q is the set of homotopy classes of loops based at q: $\pi_1(V, q)$.

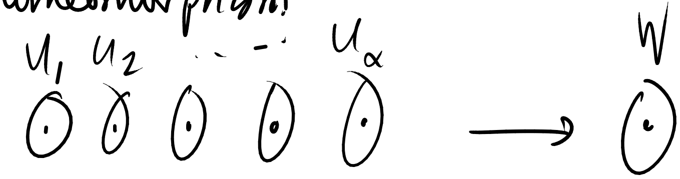
The group operation is concatenation of loops

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \gamma_2(2t-1) & 0 \le t \le 1 \end{cases}$$



Assuming V is connected, fundamental groups based at different points are isomorphic (but not canonically so).

Covering space of V : $F : U \to V$ s.t. F is onto, and $\forall p \in V$, $\exists$ open nbhd W of p s.t. $F^{-1}(W)$ consists of disjoint open sets $\{U_\alpha\}_{\alpha \in A}$ s.t. $F|_{U_\alpha} : U_\alpha \to W$ is a homeomorphism

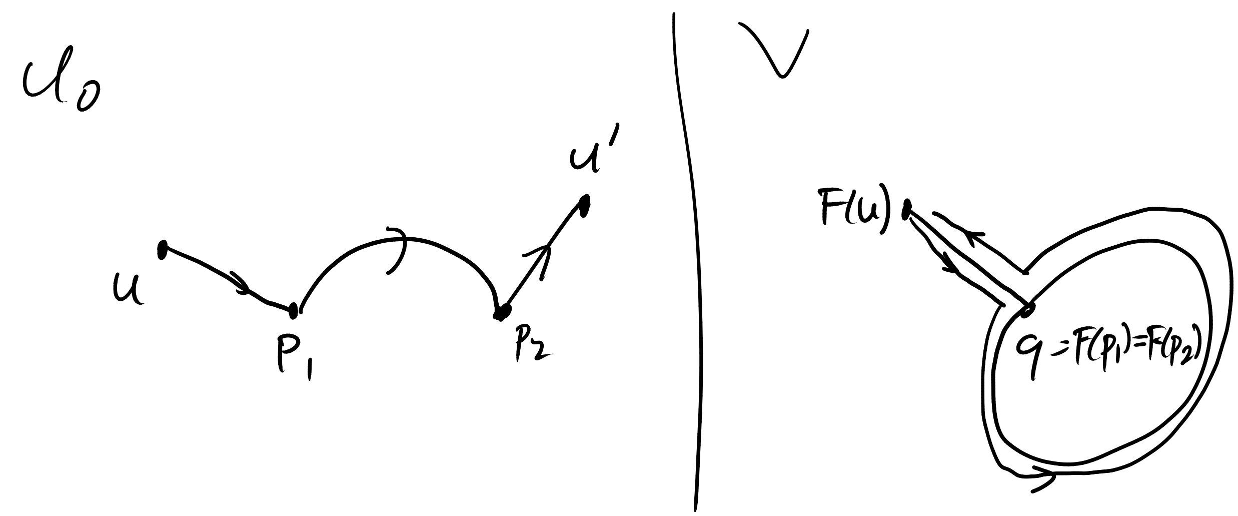


Path lifting property of covering spaces:
Given $\gamma : [0,1] \to V$ path and $p \in U$ s.t. $F(p) = \gamma(0)$, there is a unique path $\tilde{\gamma} : [0,1] \to U$ such that $\tilde{\gamma}(0) = p$ and $F(\tilde{\gamma}(t)) = \gamma(t)$.

Isomorphism of covering spaces: $F_1 : U_1 \to V$, $F_2 : U_2 \to V$
$G : U_1 \to U_2$ homeomorphism such that $F_2 \circ G = F_1$

$$\left(\begin{tikzcd} U_1 \arrow[r, "G"] \arrow[dr, "F_1"'] & U_2 \arrow[d, "F_2"] \\ & V \end{tikzcd} \text{ commutes} \right)$$

Universal covering $F_0 : U_0 \to V$ s.t. U_0 is simply connected
$\pi_1(U_0, q) = \{e\}$.
$\pi_1(V, q)$ acts on U_0 because of the path lifting property.

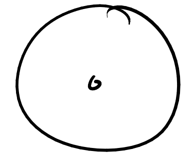


Fact: $\left\{ \begin{array}{l} \text{isomorphism} \\ \text{classes of connected} \\ \text{coverings } F : U \to V \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{conjugacy classes} \\ \text{of subgroups} \\ \quad H \subseteq \pi_1(V, q) \end{array} \right\}$

$F : U_0 / H \to V \quad \longleftarrow \quad H$

$F : U \to V \quad \longrightarrow \quad$ stabilizer of a point.

<u>Example</u>: $V = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$
punctured unit disk



$\pi_1(V, q) \cong \mathbb{Z}$ by winding number.

$\mathbb{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$

$F : \mathbb{H} \to V \qquad F(z) = \exp(2\pi i z) = e^{2\pi i \,\mathrm{Re}(z)} e^{-2\pi \,\mathrm{Im}(z)}$

This is a universal covering of V. The group $\pi_1(V,q) \cong \mathbb{Z}$ acts as follows:
for γ_n winding n times : $\mathbb{H} \to \mathbb{H}$
$z \mapsto z + n$ translation.

Subgroups of $\mathbb{Z}$: $N\mathbb{Z}$ for some $N \geq 0$.
Since $\mathbb{Z}$ is abelian, two subgroups are conjugate iff they are equal.

$N = 0$: Universal covering $F : \mathbb{H} \to V$
$N = 1$: Trivial covering $\mathrm{id} : V \to V$
$N \geq 2$: take $\mathbb{H}/(z \mapsto z + N)$ this quotient is also a punctured disk $\mathbb{D}_N = \{ z \mid 0 < |z| < 1 \}$
$\mathbb{H} \to \mathbb{D}_N$
$z \mapsto \exp\left(2\pi i \frac{z}{N}\right)$

The covering map $\mathbb{D}_N \to V \qquad (V = \mathbb{D}_1)$
$w \mapsto w^N$
The degree of the covering is $N. = |\mathbb{Z}/N\mathbb{Z}|$

Let $F: U \to V$ be a connected covering.

If F corresponds to $H \leq \pi_1(V,q)$, then the degree of F is the index of H in $\pi_1(V,q)$. Suppose this is finite, call it d.

The fiber $F^{-1}(q)$ consists of d points $\{x_1, \dots, x_d\}$

each loop γ based at q can be lifted in d ways to paths $\tilde{\gamma}_1, \dots, \tilde{\gamma}_d$ such that $\tilde{\gamma}_i(0) = x_i$

Now consider the endpoints $\tilde{\gamma}_i(1)$. Thus must also lie in $F^{-1}(q)$, so $\tilde{\gamma}_i(1) = x_j$ for some $j = 1, \dots, d$

define $\sigma(i) = j$.

This map σ is a <u>permutation</u> of the index set $\{1, \dots, d\}$

Doing this for all loops, we get a function

$$\rho : \pi_1(V,q) \to S_d$$

where S_d is the set of permutations of $\{1, \dots, d\}$

This map ρ is called the <u>monodromy</u> representation of the covering $F: U \to V$

<u>Def</u> a subgroup $H \subseteq S_d$ is called <u>transitive</u> if $\forall\, i,j\ \exists\, \sigma \in H$ s.t. $\sigma(i) = j$

<u>Lemma</u> If $F: U \to V$ is <u>connect</u> covering, then the image of $\rho : \pi_1(V,q) \to S_d$ is a transitive subgroup.

<u>Proof</u> Pick a path $x_i \to x_j$, push it down to a loop, lift it, get what you started with.