Atlases: X a topological space.
$\varphi_\alpha : U_\alpha \to V_\alpha$ complex charts $\left(\begin{array}{l} U_\alpha \subset X \text{ open} \\ V_\alpha \subset \mathbb{C} \text{ open} \\ \varphi_\alpha \text{ homeomorphism} \end{array} \right)$

Def: An atlas of complex charts on X is a collection $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in A}$ of complex charts that are

- pairwise compatible: $\varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic where defined
- cover X: $X = \bigcup_{\alpha \in A} U_\alpha$

Remark: An atlas gives us at least one complex coordinate near any point $x \in X$. Two such coordinates are related by holomorphic coordinate change.

In $U_\alpha \cap U_\beta$ we have $z = \varphi_\alpha(x)$, $w = \varphi_\beta(x)$

$w = f(z)$ where $f = \varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic.

Given an atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha\}$ it is often convenient to add more charts to $\mathcal{A}$, so long as they are compatible with the charts we already have.

Eg. If $V \subset \mathbb{C}$ is an open set, then $\{id : V \to V\}$ is an atlas with one chart. Another atlas is

$\{ id_U : U \to U \mid U \subset V \text{ open} \}$ which has uncountably many charts.

Adding compatible charts does not really change the intrinsic "complex structure" of X.

<u>Def</u> Two atlases $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha\}$, $\mathcal{B} = \{\varphi'_\beta : U'_\beta \to V'_\beta\}$

are <u>equivalent</u> if every chart of $\mathcal{A}$ is compatible w/ every chart of $\mathcal{B}$.
That is, $(\forall \alpha, \beta)$ $\varphi_\alpha : U_\alpha \to V_\alpha$ is compatible with $\varphi'_\beta : U'_\beta \to V'_\beta$

This condition is equivalent to saying that $\mathcal{A} \cup \mathcal{B}$ is an atlas.

<u>Def</u> A <u>complex structure</u> (or <u>Riemann surface structure</u>) on X is an equivalence class of atlases on X.

If $\mathcal{A}$ is any atlas, we may define

$$\mathrm{Maximal}(\mathcal{A}) = \left\{ \varphi : U \to V \;\middle|\; \begin{array}{l} \varphi : U \to V \text{ is a chart on } X \\ \text{that is compatible with} \\ \text{every chart in } \mathcal{A} \end{array} \right\}$$

<u>Observations</u>:
- $\mathcal{B}$ is equivalent to $\mathcal{A} \iff \mathcal{B} \subset \mathrm{Maximal}(\mathcal{A})$
  $\iff \mathrm{Maximal}(\mathcal{B}) = \mathrm{Maximal}(\mathcal{A})$
- $\mathcal{A}$ is equivalent to $\mathrm{Maximal}(\mathcal{A})$, etc...

<u>Def</u> An atlas of the form $\mathrm{Maximal}(\mathcal{A})$ is called a <u>maximal atlas</u>. (They are maximal elements of the set of atlases partially ordered by inclusion).

Every equivalence class of atlases contains a unique maximal atlas. So a complex structure is specified by and specifies uniquely a maximal atlas.

We usually construct Riemann surfaces by giving convenient "small" atlases. But when we prove general theorems we may want to work with maximal atlases.

<u>Def</u> A <u>Riemann surface</u> is a second countable Hausdorff topological space X, together with a complex structure (equiv class of atlases or max atlas). We will essentially always also assume X is connected.

<u>Variants</u>: Changing the definition of charts and compatibility leads to related notions.

<u>Smooth n-manifolds</u>: · Chart: $\varphi : U \to V$ where
$U \subset X$ open
$V \subset \mathbb{R}^n$ open
φ homeo.

Compatibility: $\varphi_1 : U_1 \to V_1$, $\varphi_2 : U_2 \to V_2$
require $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$

<u>and</u> $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$

to be of differentiability class C^∞.

Recall: for $V \subset \mathbb{R}^n$ open $C^\infty(V) = \left\{ f : V \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ has continuous} \\ \text{partial derivatives} \\ \text{of all order} \end{array} \right\}$

<u>Complex manifolds</u>: Chart $\varphi : U \to V$ $V \subset \mathbb{C}^n$ open
compatibility $\varphi_2 \circ \varphi_1^{-1}$ holomorphic in each variable