Quotients of Riemann surfaces by group actions.

Group actions: let G be a group and X a Riemann Surf.
An action is a map $G \times X \to X$
$$(g, p) \mapsto g.p$$
such that (a) $e \in G$ identity elt $\quad e.p = p$
(b) $g, h \in G \Rightarrow (gh).p = g.(h.p)$

The action is <u>holomorphic</u> if $\forall g \in G$, $p \mapsto g.p$ is a holomorphic map.

The action is <u>effective</u> if $(\forall p \in X \;\; g.p = p) \Rightarrow g = e$
(ie., no element of G acts by identity map)

The <u>orbit</u> through $p \in X$ is $G \cdot p = \{ g.p \mid g \in G \}$.

The <u>stabilizer</u> of $p \in X$ is $G_p = \{ g \in G \mid g.p = p \}$
(<u>isotropy subgroup</u>)

The <u>quotient space</u> X/G is the set of orbits. We topologize it as a quotient space. We want to put a complex structure on X/G, making it a Riemann surface.

Analysis of stabilizers:
<u>Proposition</u>: G acting on X holomorphically and effectively. $p \in X$
Suppose stabilizer G_p is finite. then G_p is a finite cyclic group.

<u>Proof</u> let z be a local coordinate centered at $p \in X$.
let $g \in G_p$, write $g(z) = \sum_{n=1}^{\infty} a_n(g) z^n$.

No constant term since $g(p)=p$, and $a_1(g) \neq 0$ since g has multiplicity 1 at p (as at all points).

Define a map $a_1 : G_p \to \mathbb{C}^\times$ $\qquad a_1(g) =$ coeff of z in $g(z) = \sum_{n=1}^{\infty} a_n(g) z^n$

We claim a_1 is a homomorphism.

$g, h \in G_p$

$$(gh)(z) = g(h(z)) = g\left(a_1(h) z + \sum_{n \geq 2} a_n(h) z^n\right)$$

$$= a_1(g) a_1(h) z + (\text{higher order terms})$$

so $a_1(gh) = a_1(g) a_1(h)$.

We claim a_1 is injective. Since the only finite subgroups of $\mathbb{C}^\times$ are finite cyclic, this will complete the proof.

Suppose $a_1(g) = 1$ thus $g(z) = z + \sum_{n \geq 2} a_n(g) z^n$.

Let $n_0 = \min \{ n \mid {n \geq 2 \atop a_n(g) \neq 0} \}$ so $g(z) = z + a_{n_0}(g) z^{n_0} + (\text{higher})$

$$g^k(z) = z + k a_{n_0}(g) z^{n_0} + (\text{higher})$$

for some $k > 0$ $g^k(z) = z$, so $k a_{n_0}(g) = 0$, but then $a_{n_0}(g) = 0$, contradicting definition of n_0. So in fact $g(z) = z$.
Since g is identity in a neighborhood of p, it is identity on all of X.
Since the action is effective, $g = e$.

Prop If G is finite,
The set of $p \in X$ such that $G_p \neq \{e\}$ is discrete.

Proof: suppose $\{p_n\}_{n\geq 1}$ is a sequence of points with nontrivial stabilizer.

After passing to a subsequence, we may assume all p_n are fixed by a single $g \in G$ (using G finite).

Thus $G = id$ on a subset of X having a limit point in X, so $g = id$.

To define a Riemann surface structure on X/G, we need a nice neighborhood of points with nontrivial stabilizers.

Proposition: G a finite group, acting holomorphically and effectively on X. Let $p \in X$. There is a open neighborhood $U \ni p$ such that

(a) U is invariant under G_p: $(\forall g \in G_p)(g \cdot U = U)$

(b) $U \cap g(U) = \emptyset$ if $g \notin G_p$.

(c) $\alpha : U/G_p \to X/G$ is a homeo onto an open set in X/G

(d) no point in U is fixed by any element of G_p, except for p itself.

Proof: Let $G \setminus G_p = \{g_1, \dots, g_n\}$ For each i, choose

open $V_i \ni p$ and $W_i \ni g_i \cdot p$ such that $V_i \cap W_i = \emptyset$.

Now $g_i^{-1} W_i$ is a neighborhood of p.

Let $R_i := V_i \cap (g_i^{-1} W_i)$, $\quad R := \bigcap_i R_i$

$U = \bigcap_{g \in G_p} g \cdot R$

R_i, R, and U are open sets containing p, and clearly $g \cdot U = U$ if $g \in G_p$ satisfying (a)

For (b): $R_i \cap (g_i R_i) \subset V_i \cap W_i \neq \emptyset$ so $R \cap (g_i R) = \emptyset$ and $U \cap (g_i U) \notin \emptyset$.

For (c): U/G_p makes sense by (a); $\alpha: U/G_p \to X/G$ is 1-1 by (b). By general nonsense of quotient maps, α is continuous and an open mapping. Hence it is a homeo onto its image.

For (d): Using discreteness of points with nontrivial stabilizer, we can arrange that V_i has no such points other than p, and then U will as well.

To define charts on X/G, define them on U/G_p and transport them to X/G via $\alpha: U/G_p \to X/G$.

Basic idea: holomorphic function on U/G_p = G_p-invariant holomorphic function on U.

We might as well assume U is small enough to be the domain of chart $\varphi: U \to V \subset \mathbb{C}$ on X.

Consider $\psi(x) = \prod_{g \in G_p} (\varphi \circ g)(x)$. This function is holomorphic and G_p-invariant.

This function has multiplicity $m = |G_p|$ at p. indeed if z is a local coord centered at p, and g_1 is a generator of the cyclic group G_p then

$$\psi(z) = \prod_{i=0}^{m-1} g_1^i(z) = \prod_{i=0}^{m-1} \left(a_1^i z + \text{h.o.t.} \right) = \left(\prod_{i=0}^{m-1} a_1^i\right) z^m + \text{h.o.t}$$

Since $\psi : U \to \mathbb{C}$ is G_p invariant, it descends to a function $\bar{\Psi} : U/G_p \to \mathbb{C}$. This map is open

$\bar{\Psi}$ is 1-1: Indeed, since ψ has multiplicity m at p, If $w \in \operatorname{Im}\psi$ and $w \neq \psi(p)$, then $\psi^{-1}(w)$ has m points. on the other hand, if $\psi(q) = w$, then

$$\underset{\uparrow}{G_p q} \subset \underset{\uparrow}{\psi^{-1}(w)}$$

both sets have size m, so they are equal.

Thus $\psi(q) = \psi(q') \iff q' \in G_p q$.

Now we get a chart by $\alpha(U/G_p) \xrightarrow{\alpha^{-1}} U/G_p \xrightarrow{\bar{\Psi}} V \subset \mathbb{C}$

Note if $G_p = \{e\}$ and $m=1$, $\alpha : U \hookrightarrow X/G$ is an embedding, and we are just using a chart on X.

<u>Theorem</u> G finite group acting holomorphically and effectively on X then the above prescription defines a complex atlas on X/G the map $\pi : X \to X/G$ is holomorphic $\deg \pi = |G|$ and $\operatorname{mult}_p(\pi) = |G_p|$.

<u>Addendum</u>: For each branch point $y \in X/G$, let $\{x_1, \ldots, x_s\} = \pi^{-1}(y) = G \cdot x_i$ then the stabilizer subgroups G_{x_i} are conjugate in G. Let $r = |G_{x_i}|$ (independent of i). Thus $|G| = r\,|\pi^{-1}(y)|$ and $|\pi^{-1}(y)| = \frac{|G|}{r} = |G \cdot x_i|$

Applying Riemann-Hurwitz: if $y_1, \dots, y_k \in X/G$ are branch points with $\frac{|G|}{r_i}$ preimages of multiplicity r_i, then

$$2g(X)-2 = |G|\left(2g(X/G)-2\right) + \sum_{i=1}^{k} \frac{|G|}{r_i}(r_i - 1)$$

$$= |G|\left[2g(X/G)-2 + \sum_{i=1}^{k}\left(1-\frac{1}{r_i}\right)\right]$$

Using this relation, one can try to classify various kinds of groupactions on Riemann surfaces of low genus.

Infinite groups:

<u>Definition</u>: Let G act on a Hausdorff space X. This action is <u>properly discontinuous</u> if $\forall$ pairs $p, q \in X$ $\exists$ open U, V, $p \in U$, $q \in V$ such that

$$\{g \in G \mid (g \cdot U) \cap V \neq \emptyset\} \text{ is finite.}$$

Under this extra condition, the same analysis as before works, and we can define X/G as a Riemann surface.

<u>Key example</u>: $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$

$$\operatorname{Aut}(\mathbb{H}) = PSL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{matrix} a,b,c,d \in \mathbb{R} \\ ad-bc=1 \end{matrix} \right\} \Big/ \langle \pm I \rangle$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

A compact R.S. of genus ≥ 2 is $\Gamma \backslash \mathbb{H}$ where $\Gamma \subset PSL(2,\mathbb{R})$ act properly discontinuously and fixed point freely.