More Resolutions: Monomial singularities

A <u>monomial singularity</u> is one modeled on $z^n = w^m$

<u>Def</u> Let $f(x,y)$ be a polynomial such that $f(0) = \frac{\partial f}{\partial x}(0) = \frac{\partial f}{\partial y}(0) = 0$. Thus $X = V(f)$ is a plane curve with a singularity at the origin.
We say the origin is an <u>(n,m)-monomial singularity</u> if there are holomorphic functions $g(x,y)$, $h(x,y)$ defined in a neighborhood of 0 such that

$$g(0)=0,\ h(0)=0,\ \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix} \text{ is nonsingular at } 0$$

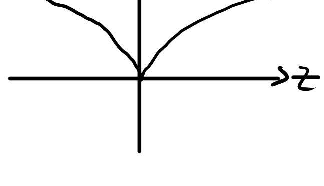
and $$f(x,y) = g(x,y)^n - h(x,y)^m.$$

In otherwords, $z = g(x,y)$ and $w = h(x,y)$ define local coordinates on $\mathbb{C}^2$ near zero, and $f = z^n - w^m$ in these coordinates.

To understand such singularities, it suffices to consider the model case $\{z^n = w^m\} \subset \mathbb{C}^2$.

<u>Note</u>: $(2,2)$-monomial sing. = node
$$z^2 - w^2 = (z-w)(z+w)$$

<u>Next case</u>: $(2,3)$-monomial sing is called <u>cusp</u>.
$z^2 = w^3$



Wm. Niele 1657
(computed arc length)

$X = \{(z,w) \mid z^2 = w^3\}$

Unlike the node, if we consider $X \setminus \{(0,0)\}$, we get something connected!

Note the obvious parametrization
$$\mathbb{C} \to X$$
$$t \mapsto (t^3, t^2)$$

This map is actually a homeomorphism, but <u>not</u> an isomorphism of algebraic varieties (whatever that means)

The inverse is
$$\varphi : X \setminus \{(0,0)\} \longrightarrow \mathbb{C} \setminus \{0\}$$
$$(z,w) \longmapsto z w^{-1}$$

Indeed: $t \to (t^3, t^2) \longrightarrow t^3 (t^2)^{-1} = t$

$$(z,w) \to z w^{-1} \longrightarrow ((zw^{-1})^3, (zw^{-1})^2)$$
$$= (z^3 w^{-3}, z^2 w^{-2})$$
$$= (z, w)$$

we can use φ as a hole chart and fill this hole. The result is $\tilde{X} = X \setminus \{(0,0)\} \cup \{pt\}$. infact the result is isomorphic to $\mathbb{C}$. Indeed, the hole chart is an isomorphism of $X \setminus \{(0,0)\}$ with $\mathbb{C} \setminus \{0\}$, so filling the holes gives isomorphic surfaces.

Doing this locally near any cusp replaces a neighborhood of the cusp with a (smooth) disk.

The same procedure works if $\gcd(n,m) = 1$:

$X = \{z^n = w^m\}$

parametrization: $t \longrightarrow (t^m, t^n)$

Since $\gcd(n,m) = 1$, can find $a, b \in \mathbb{Z}$ s.t. $an + bm = 1$

hole chart
$$X \setminus \{(0,0)\} \longrightarrow \mathbb{C} \setminus \{0\}$$
$$(z,w) \longrightarrow z^b w^a$$

Filling this hole yields $\tilde{X}$, which is isomorphic to $\mathbb{C}$

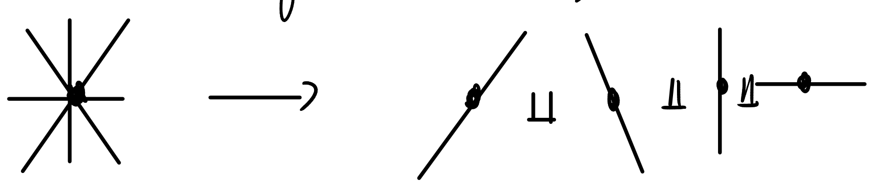
Can also do this locally near any (n,m)-monomial singularity, replaces neighborhood of singularity with a smooth disk.

<u>Case</u> $n=m$ $\qquad z^n = w^n$

$$z^n - w^n = \prod_{i=0}^{n-1} (z - \zeta_n^i w) \quad , \quad \zeta_n = e^{2\pi i/n}$$

This is like the node but with more than two factors.

$X \setminus \{(0,0)\}$ now has n components, each of which has a hole. Filling all these holes seperates the n branches of the solution set.



<u>General case</u>: $\quad k := \gcd(n,m) \qquad n = ka \quad m = kb$

$$z^n - w^m = (z^a)^k - (w^b)^k = \prod_{i=0}^{k-1} (z^a - \zeta_k^i w^b)$$

Thus there are k seperate factors, each of which looks like the relatively prime case.

<u>Thm</u>: Suppose $X = \{ (x,y) \mid f(x,y) = 0 \}$ has an (n,m)-monomial singularity at 0. Then there is a neighborhood U of 0 such that $(X \cap U) \setminus \{(0,0)\}$ has $k = \gcd(n,m)$ connected components, each of which has a hole at 0. Filling these holes replaces $X \cap U$ with k disks.

Names: $(2,2)$ = node, $(2,3)$ = cusp, $(2,4)$ = tacnode
$(2,m) = A_{m-1}$ $(3,3) = D_4$, $(3,4) = E_6$
$(3,5) = E_8$ etc.

Cyclic Riemann Surfaces

$h(x)$ polynomial
(Don't assume distinct roots)

$$X = \{(x,y) \mid y^d = h(x)\}$$

If a is a repeated root of $h(x)$, X has a monomial singularity at $(a,0)$

$h(x) = (x-a)^n g(x)$ where $g(a) \neq 0$

pick $\sqrt[n]{g(x)}$ defined near a. $h(x) = \left((x-a)\sqrt[n]{g(x)}\right)^n = w^n$
$w = (x-a)\sqrt[n]{g(x)}$ is valid near $x=a$.

So in (w,y) coordinates, we see X has a (d,n)-monomial singularity near $(a,0)$. Resolve it as above.

Similar to the case of hyperelliptic curves, we can also fill in the holes at ∞:

let $\deg h(x) = k$ write $k = d\ell - \varepsilon$ $0 \le \varepsilon < d$
let $z = \frac{1}{x}$ $y^d = h\left(\frac{1}{z}\right)$

$$z^{d\ell} y^d = z^{d\ell} h\left(\tfrac{1}{z}\right) = k(z) \text{ polynomial}$$

$$w = z^{\ell} y = \frac{y}{x^{\ell}} \qquad w^d = k(z)$$

Do the resolution process with $Y = \{(z,w) \mid w^d = k(z)\}$
the glue $X \cap \{x \neq 0\}$ to $Y \cap \{z \neq 0\}$.