Holes / Punctures.

$X \longrightarrow X \setminus \{p\}$ has a hole/puncture.

Can try to reverse this process.

<u>Def</u>: A <u>hole chart</u> on a Riemann surface X is a chart $\phi : U \to V$ on X such that V contains an open punctured disk $D_0 = \{ z \mid 0 < |z - z_0| < \varepsilon \} \subset V$ and such that $\overline{\varphi^{-1}(D_0)} \subset U$ and

$$\varphi\left(\overline{\varphi^{-1}(D_0)} \right) = \{ z \mid 0 < |z - z_0| \leq \varepsilon \}.$$

This last condition excludes e.g.
$X = \mathbb{C}$, $\underline{U = \mathbb{C}}$, $V = \mathbb{C}$, $\varphi = id$. $D_0 \subset V$,
but $\overline{\varphi^{-1}(D_0)}$ contains z_0, so X doesn't really have a hole.

Given a hole chart $\varphi : U \to V$ on X, we can define another R.S. by "filling in the hole." Let $D_0 = \{ 0 < |z - z_0| < \varepsilon \} \subset V$ and let $D = \{ |z - z_0| < \varepsilon \}$. Now glue D to X by identifying $D_0 \subset D$ with $\varphi^{-1}(D_0) \subset X$.

$Z = X \amalg D / \phi$. The condition on the closure of D guarantees that Z is Hausdorff.

Examples: • $X = \mathbb{C}$ $\quad U = \mathbb{C} \setminus \{0\}$ $\quad V = \mathbb{C} \setminus \{0\}$

$$\varphi : U \to V \quad \text{is a hole chart.}$$
$$z \mapsto 1/z$$

filling this hole yields $Z = \hat{\mathbb{C}}$

• $X = \{(x,y) \mid y^2 = h(x)\} \subset \mathbb{C}^2$ $\quad h$ degree $2g+1$ or $2g+2$ with distinct roots.

If h has degree $2g+1$, consider $\dfrac{y}{x^{g+1}}$ for $|x| > C >> 1$

$$\left(\frac{y}{x^{g+1}}\right)^2 = \frac{h(x)}{x^{2g+2}} = \frac{a_{2g+1}}{x} + \frac{a_{2g}}{x^2} + \cdots + \frac{a_0}{x^{2g+2}}$$

For $|x| > C >> 1$, the map $(x,y) \to \dfrac{y}{x^{g+1}}$ is a hole chart

If $\deg h = 2g+2$, $\left(\dfrac{y}{x^{g+1}}\right)^2 = \dfrac{h(x)}{x^{2g+2}} = a_{2g+2} + \dfrac{a_{2g+1}}{x} + \cdots$

As $x \to \infty$, $\dfrac{y}{x^{g+1}}$ approaches $\pm\sqrt{a_{2g+2}}$

There are two hole charts on X in the region $|x| > C >> 1$, one for each square root of a_{2g+2}.

In both cases, plugging the hole is another way to obtain the compact hyperelliptic curves we constructed before.

We can use this hole-plugging idea to get many more Riemann surfaces.

Resolution of singularities for curves:

General idea: An algebraic curve may fail to define a Riemann surface if it has singularities (is not smooth). But the singularities are a discrete set of points. Delete these points, check that the result has holes in the above sense, and then plug the holes, obtaining a (smooth) Riemann surface.

Plane curves with nodes: Affine case: $X = \{(x,y) \mid f(x,y)=0\} \subset \mathbb{C}^2$

We allow f to be singular $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ may both vanish at some points of X.

<u>Def</u> $p \in X$ is a <u>node</u> if p is a singular point $\frac{\partial f}{\partial x}(p) = 0 = \frac{\partial f}{\partial y}(p)$

and the matrix of second derivatives (Hessian)

$$\mathrm{Hess}(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \text{ is nonsingular,}$$

i.e. $f_{xx}(p) f_{yy}(p) - (f_{xy}(p))^2 \neq 0$.

(Same definition as a nondegenerate critical point in Calc III, but we're in the holomorphic setting)

$$f = \left.\begin{matrix} f(p) + f_x(p)(x-x_0) + f_y(p)(y-y_0) + \end{matrix}\right\} \text{ these all vanish } p \in X \text{ singular.}$$

$$+ \frac{f_{xx}(p)}{2}(x-x_0)^2 + f_{xy}(p)(x-x_0)(y-y_0) + \frac{f_{yy}(p)}{2}(y-y_0)$$

+ higher order terms

The nondegeneracy of Hess(f) implies that the quadratic part

$$\frac{f_{xx}(p)}{2}(x-x_0)^2 + f_{xy}(p)(x-x_0)(y-y_0) + \frac{f_{yy}(p)}{2}(y-y_0)$$

$$= \ell_1(x-x_0, y-y_0)\,\ell_2(x-x_0, y-y_0)$$

where ℓ_1 and ℓ_2 are distinct linear homog. polynomials.

Lemma

If $f(x,y) = \ell_1(x-x_0, y-y_0)\,\ell_2(x-x_0, y-y_0) + (\text{terms of order} \geq 3)$

Then as a powerseries $f(x,y) = g(x,y)\,h(x,y)$ where

$$g(x,y) = \ell_1(x-x_0, y-y_0) + (\text{terms of order} \geq 2)$$

$$h(x,y) = \ell_2(x-x_0, y-y_0) + (\text{terms of order} \geq 2)$$

And g, h are convergent power series.

Remark: This is a special case of Hensel's Lemma, and also the Morse Lemma.

Proof: Let us use coordinates $z = \ell_1(x-x_0, y-y_0)$
$w = \ell_2(x-x_0, y-y_0)$

Thus $f = zw + \sum_{i=3}^{\infty} f_i$ where f_i is homogeneous of degree i in z, w

Posit $g = z + \sum_{i=2}^{\infty} g_i$ and $h = w + \sum_{i=2}^{\infty} h_i$

g_i, h_i homog of degree i in z, w.

Want $f = gh$

We show that g_i and h_i can be solved for recursively

$$zw + \sum_{i \geq 3} f_i = f = gh = \left(z + \sum_{i \geq 2} g_i\right)\left(w + \sum_{i \geq 2} h_i\right)$$

$$zw + \sum_{i \geq 3} f_i = zw + \sum_{i \geq 3}\left(z h_{i-1} + w g_{i-1} + \sum_{j=2}^{i-2} g_j h_{i-j}\right)$$

So need to solve for all $i \geq 3$
$$f_i = z h_{i-1} + w g_{i-1} + \sum_{j=2}^{i-2} g_j h_{i-j}$$

Base case $i = 3$ $\quad f_3 = z h_2 + w g_2$

Possible solution $h_2 = \dfrac{(\text{terms not involving } w)}{z}$ $\quad g_2 = \dfrac{\text{terms involving } w}{w}$

Induction $\quad f_i = z h_{i-1} + w g_{i-1} + \sum_{j=2}^{i-2} g_i h_{i-j}$

$$\underbrace{f_i - \sum_{j=2}^{i-2} g_i h_{i-j}}_{\text{already fixed.}} = z h_{i-1} + w g_{i-1} \to \text{same trick works.}$$

$\boxtimes$

Thus, in some neighborhood U of p, $f = gh$.

This $X \cap U = \{q \in U \mid f(q) = 0\} = X_g \cup X_h$

where $\quad X_g = \{q \in U \mid g(q) = 0\}$
$\quad X_h = \{q \in U \mid h(q) = 0\}$

since $\frac{\partial g}{\partial z}(p) \neq 0$ and $\frac{\partial h}{\partial w}(p) \neq 0$, X_g and X_h are smooth!

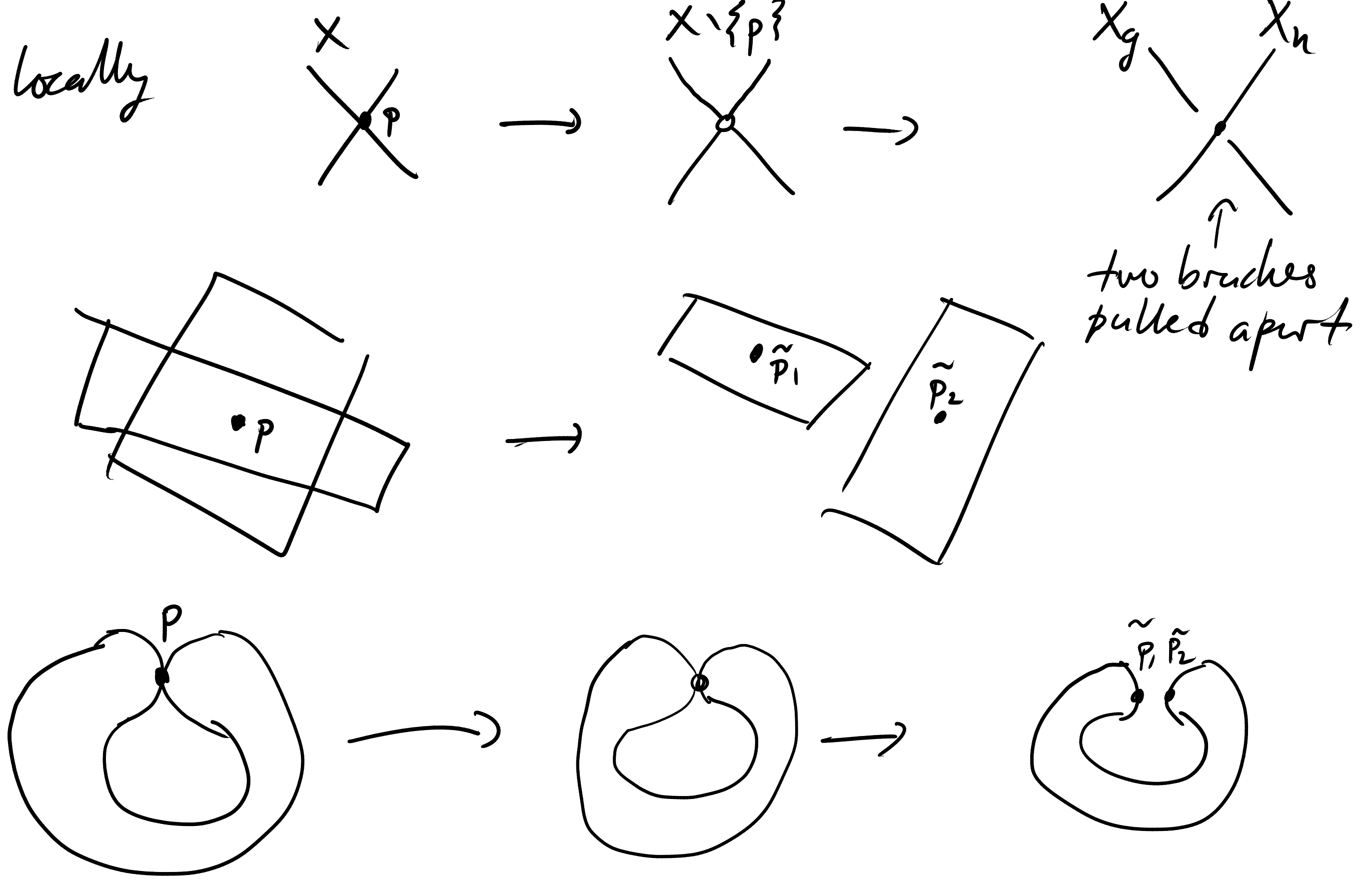
and $X_g \cap X_h = \{p\}$.

Consider $(X \cap U)\setminus\{p\} = (X_g \setminus \{p\}) \cup (X_h \setminus \{p\})$

Thus $X\setminus\{p\}$ has two hole charts near p: $X_g\setminus\{p\}$ and $X_h\setminus\{p\}$

Fill these holes yields $\tilde{X}$: p is replaced by two points

locally



This construction is local, so it works just as well with projective curves. If a projective curve $\{F(x,y,z)=0\} \subset \mathbb{CP}^2$ is smooth except for some nodes, and we resolve all the nodes, we obtain a compact Riemann surface.