Maps Between complex tori

<u>Prop</u>: Every map $F : \mathbb{C}/L_1 \to \mathbb{C}/L_2$

is of the form $F(z) = \gamma z + c$
where $\gamma, c \in \mathbb{C}$ and $\gamma L_1 \subseteq L_2$

Degree $= |L_2/\gamma L_1| = [\gamma L_1 : L_2]$

Never ramified $\quad \chi(\mathbb{C}/L_1) = d\,\chi(\mathbb{C}/L_1) - \sum_p [\mathrm{mult}_p - 1]$

$$0 = 0 - \sum_p [\mathrm{mult}_p - 1]$$

Isomorphisms: $\iff \gamma L_1 = L_2$

$\mathbb{C}/L_1 \cong \mathbb{C}/L_2 \iff \exists\, \gamma \in \mathbb{C}\setminus\{0\}$ s.t. $\gamma L_1 = L_2$

<u>Problem</u>: Classify complex tori up to isomorphism
equivalent to classifying lattices $L \subset \mathbb{C}$ up to
multiplication by $\gamma \in \mathbb{C}\setminus\{0\}$.

Not so trivial: make it easier by introducing
<u>extra data</u>.

Based lattices := triples (L, ω_1, ω_2) such that
ω_1, ω_2 indep over $\mathbb{R}$, and $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

Oriented based lattices = (L, ω_1, ω_2) such that
additionally ω_1, ω_2 is positively oriented basis.

Equivalence relation $(L, \omega_1, \omega_2) \sim (L', \omega_1', \omega_2')$

$$\Updownarrow$$

$\exists \gamma \in \mathbb{C}^*$ s.t. $\gamma L = L'$, $\gamma\omega_1 = \omega_1'$, $\gamma\omega_2 = \omega_2'$

<u>Prop</u> $\{\text{based lattices}\}/\sim \;\longleftrightarrow\; \mathbb{C} \setminus \mathbb{R} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau \neq 0\}$

$\left\{\begin{matrix}\text{oriented based} \\ \text{lattices}\end{matrix}\right\}/\sim \;\longleftrightarrow\; \mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$

$$(L, \omega_1, \omega_2) \longmapsto \omega_2/\omega_1$$

$$(\mathbb{Z} + \mathbb{Z}\tau, 1, \tau) \longleftarrow\!\shortmid \tau$$

It remains to "forget" the extra data of a basis

$$\left\{\begin{matrix}\text{oriented based} \\ \text{lattices}\end{matrix}\right\}/\sim \;\xrightarrow{\pi}\; \{\text{Lattices}\}/\sim$$

The fiber $\pi^{-1}([L])$ may be identified with the set of all oriended basis of L.

<u>Prop</u> For a given lattice $L \subset \mathbb{C}$, the set of all oriented bases of L is a principal $SL(2,\mathbb{Z})$-set ($SL(2,\mathbb{Z})$-torsor)

In fact, the group $SL(2,\mathbb{Z})$ acts on the set of oriended based lattices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (L, \omega_1, \omega_2) = (L, d\omega_1 + c\omega_2, b\omega_1 + a\omega_2)$$

This action is compatible with equivalence of based lattices
$SL(2,\mathbb{Z})$ acts on the equivalence classes of oriented based lattices.

In terms of $\tau \in \mathbb{H}$

$$\tau \longrightarrow (\mathbb{Z}+\mathbb{Z}\tau, 1, \tau) \xrightarrow{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} (\mathbb{Z}+\mathbb{Z}\tau, d+c\tau, b+a\tau)$$

$$\downarrow$$

$$\frac{a\tau+b}{c\tau+d}$$

The equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tau = \dfrac{a\tau+b}{c\tau+d}$ defines an action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$, and there is a bijection

$$\{\text{lattices}\}/\sim \longrightarrow SL(2,\mathbb{Z})\backslash\mathbb{H}$$

The quotient $SL(2,\mathbb{Z})\backslash\mathbb{H}$ actually is a Riemann surface called the modular surface (or curve)
↳ related to theory of modular forms.

Automorphisms of $\mathbb{C}/L$: these are complex numbers such that $\gamma L = L$

$$\mathrm{Aut}(L) = \{\gamma \in \mathbb{C} \mid \gamma L = L\}$$

<u>Prop</u> We have the following cases.

- $L \sim \mathbb{Z} + \mathbb{Z}i$, $\mathrm{Aut}(L) = \mu_4 = \{z \mid z^4 = 1\}$
  $= \{\pm 1, \pm i\}$

- $L \sim \mathbb{Z} + \mathbb{Z}e^{2\pi i/6}$, $\mathrm{Aut}(L) = \mu_6 = \{z \mid z^6 = 1\}$
  $= \{e^{2\pi i k/6}\}$

- $L \sim$ anything else, $\mathrm{Aut}(L) = \mu_2 = \{z \mid z^2 = 1\} = \{\pm 1\}$.

<u>Proof idea</u>: consider the shortest vectors in $L \setminus \{0\}$.