Examples of Riemann surfaces

Basic examples: curves of low degree in $\mathbb{CP}^2$

<u>Lines:</u> A line is defined by a homogeneous polynomial of degree 1

$$F(x,y,z) = ax+by+cz \qquad a,b,c \text{ not all zero.}$$

Prop any line in $\mathbb{CP}^2$ is nonsingular and isomorphic to $\mathbb{CP}^1$

<u>Pf:</u> $\nabla F = (a,b,c)$ Not zero by assumption.

Suppose $a \neq 0$. then
$$\mathbb{CP}^1 \to V(ax+by+cz)$$
$$[r:s] \longmapsto \left[\frac{-(br+cs)}{a} : r : s\right]$$
is an isomorphism.
If $b \neq 0$ or $c \neq 0$, similar formulas work. $\blacksquare$

<u>Conics:</u> defined by homog. poly. of degree 2 = "quadratic form"

$$F(x,y,z) = ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$$
Quadratic forms correspond to symmetric matrices.

$$F(x,y,z) = (x \ y \ z)\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= V^T A_F V$$

<u>Prop:</u> F is nonsingular iff A_F is invertible.

Pf: $\nabla F(x,y,z) = 2A_F\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ singular $\Leftrightarrow \exists V_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
s.t. $A_F V_0 = 0 \Leftrightarrow A_F$ not invertible.

<u>Prop</u>. All nonsingular conics are isomorphic to each other.

linear algebra: Any nonsingular symmetric matrix A can be written as $A = B^T B$. B invertible

(A symmetric $\Rightarrow$ A diagonalizable $\Rightarrow$ $A = O^T D O$ where O orthogonal $O^T = O^{-1}$, D diagonal.

Pick a square root $\sqrt{D}$. Then $A = O^T \sqrt{D}\sqrt{D}\, O = O^T \sqrt{D}^T \sqrt{D}\, O$
$= (\sqrt{D}\, O)^T (\sqrt{D}\, O)$. Let $B = \sqrt{D}\, O$.)

Now, we can use B to change coordinates $W = BV$

Then $V^T A V = V^T B^T B V = (BV)^T (BV) = W^T W$

So in the coordinates $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = W$, $F = (x')^2 + (y')^2 + (z')^2$.

So $V(F) \cong V(x^2+y^2+z^2)$

More is true: the invertible matrix B defines an automorphism of $\mathbb{CP}^2$ that maps $V(F)$ isomorphically to $V(x^2+y^2+z^2)$ $\boxtimes$

<u>Prop</u>: A nonsingular conic is isomorphic to $\mathbb{CP}^1$

<u>Pf</u>: By previous proposition, we can check any particular conic.

Take $V(xz - y^2) \subseteq \mathbb{CP}^2$

The map $\mathbb{CP}^1 \to V(xz-y^2)$
$[r:s] \mapsto [r^2 : rs : s^2]$ is an isomorphism.

Hyperelliptic Riemann surfaces. Let $h(x)$ be a polynomial of degree $2g+1+\varepsilon$ where $\varepsilon = 0$ or 1. Assume $h(x)$ has distinct roots. Form the affine plane curve

$$X = \{(x,y) \mid y^2 = h(x)\} \qquad X \text{ is smooth.}$$

X is not compact, but we will glue on another chart to make it so.

Let $k(z) = z^{2g+2} h(\frac{1}{z})$. Since h has degree $2g+1$ or $2g+2$,
This is a polynomial.

- $h(0) \neq 0$, $h(x) = a_0 + a_1 x + \cdots \Rightarrow k(z) = a_0 z^{2g+2} + a_1 z^{2g+1} + \cdots$
  so $k(z)$ has degree $2g+2$
  $h(x)$ vanishes at $2g+1+\varepsilon$ points in $\mathbb{C}^*$
  $k(z)$ vanishes at $2g+1+\varepsilon$ points in $\mathbb{C}^*$
  if $\varepsilon = 0$, $k(z)$ also vanishes at $z=0$.

- $h(0) = 0$, $h(x) = a_1 x + \cdots \Rightarrow k(z) = a_1 z^{2g+1} \cdots$
  so $k(z)$ has degree $2g+1$
  $h(x)$ vanishes at $2g+\varepsilon$ points in $\mathbb{C}^*$
  $\Rightarrow k(z)$ vanishes at $2g+\varepsilon$ points in $\mathbb{C}^*$
  If $\varepsilon=0$, $k(z)$ also vanishes at $z=0$.

In all cases $k(z)$ has distinct roots.

Let $Y = \{(z,w) \mid w^2 = k(z)\}$

Now we glue X to Y:

$$\begin{array}{ccc}
X = \{(x,y) \mid y^2 = h(x)\} & & Y = \{(z,w) \mid w^2 = k(z)\} \\
\cup & & \cup \\
U = \{(x,y) \in X \mid x \neq 0\} & & V = \{(z,w) \in Y \mid z \neq 0\} \\
\Psi & & \Psi \\
(x,y) & \xmapsto{\ \phi\ } & (z,w) = \left(\frac{1}{x}, \frac{y}{x^{g+1}}\right) \\
(x,y) = \left(\frac{1}{z}, \frac{w}{z^{g+1}}\right) & \xleftarrow{\ \phi^{-1}\ } & (z,w)
\end{array}$$

Gluing X to Y by identifying U and V via ϕ, we obtain a Riemann surface Z.

<u>Proposition</u> Z is compact and has genus equal to g.
The function $x : X \to \mathbb{C}$ extends to a map
$\pi : Z \to \hat{\mathbb{C}}$ that has degree 2, and $2g+2$
ramification points of multiplicity 2.

<u>Proof</u>: Z is the union of $X \cap \{\|x\| \leq 1\}$ and $Y \cap \{\|z\| \leq 1\}$
which are compact.

The map $x : X \to \mathbb{C}$ clearly has degree 2: for most values of x,
$y^2 = h(x)$ has exactly two solutions for y.
The ramification points are where mult ≥ 2. These are
the points $(x_i, 0)$, where x_i is a root of $h(x) = 0$.
At all these points, $\mathrm{mult}_{(x_i,0)}(\pi : X \to \mathbb{C}) = 2$

In the other chart $z = \frac{1}{x}$ $w = \frac{y}{x^{g+1}}$ $w^2 = k(z)$
projection to x becomes projection to z.
This map is ramified at $z=0$ iff $k(0)=0$, which happens iff $\varepsilon = 0$.
If this happens, $\mathrm{mult}_{(z,w)=(0,0)}(\pi) = 2$.

In all cases, π has $2g+2$ points of multiplicity 2.
Now apply Riemann-Hurwitz formula:

$$\chi(Z) = \deg(\pi)\,\chi(\hat{\mathbb{C}}) - (2g+2)(2-1)$$
$$\chi(Z) = 2 \cdot 2 - 2g+2 = 2-2g$$

Thus Z does indeed have genus equal to g.