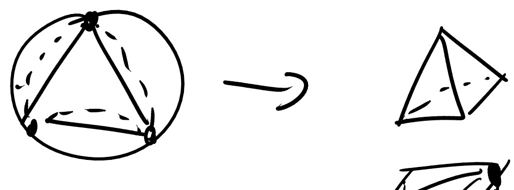
Euler characteristic and Riemann-Hurwitz formula

Euler characteristic of a triangulated surface.

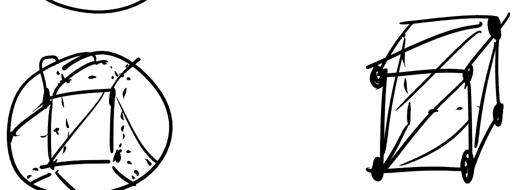
χ = #vertices − #edges + #faces

sphere



$\chi = 4-6+4=2$

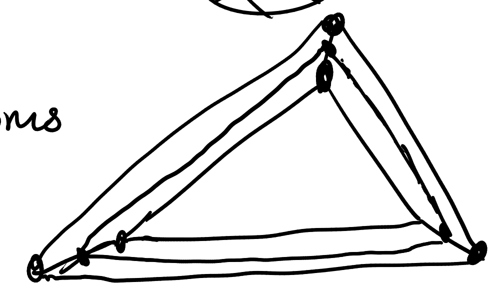
or



cube w/ faces divided

$\chi = 8-18+12=2$

Torus



9 quadrilaterals
18 edges
9 vertices

$\chi = 9-18+9=0$

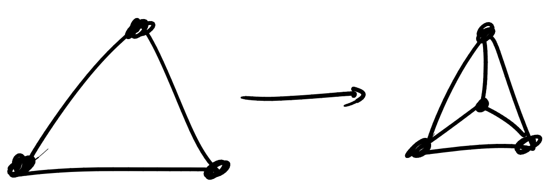
could also divide each quadrilateral into two triangles
→ 18 faces and 9 extra edges

$\chi = 18 - 27 + 9 = 0$ still.

Proposition: The Euler characteristic depends only on the homeomorphism type of the surface, not the particular triangulation.

Proof idea: Refinement of a triangulation

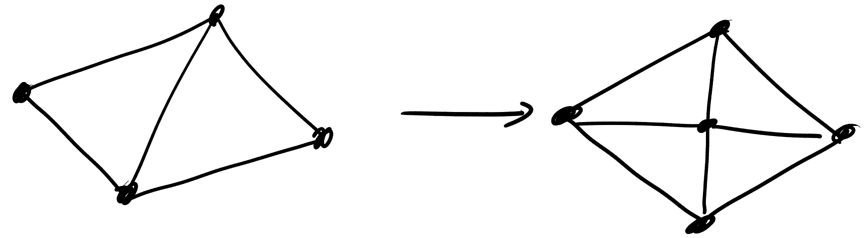
Divide triangle into 3:



$\Delta V = 1$, $\Delta E = 3$, $\Delta F = 2$

so $\Delta \chi = 1 - 3 + 2 = 0 \Rightarrow \chi$ does not change.

Divide an edge:



$\Delta V = 1 \qquad \Delta E = 3 \qquad \Delta F = 2$

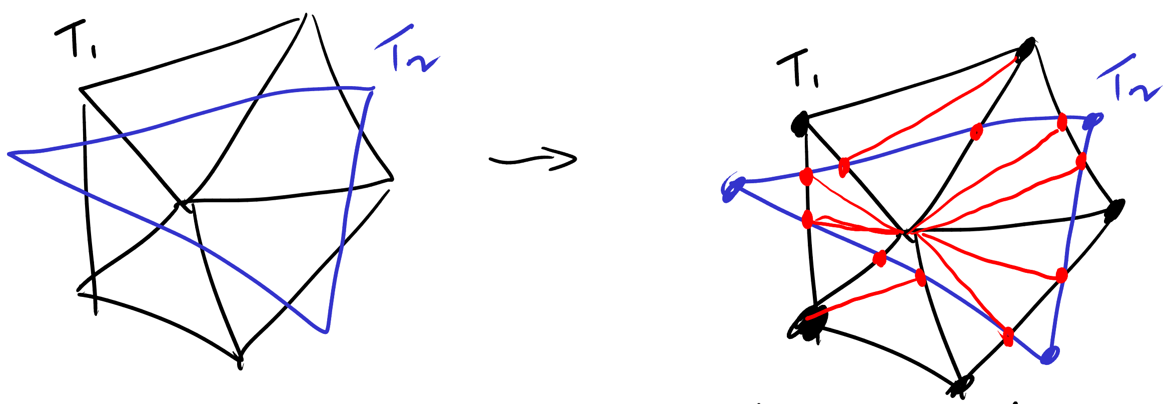
$\Delta \chi = 1 - 3 + 2 = 0$

We say that T' is a <u>refinement</u> of T if it is obtained from T by repeatedly applying the above operations.

T' refinement of $T \Rightarrow \chi(T') = \chi(T)$

<u>Lemma</u>: Any pair of triangulations T_1 and T_2 has a common refinement T'

<u>Proof Idea</u>: Superimpose the T_1 and T_2 then refine to make a triangulation



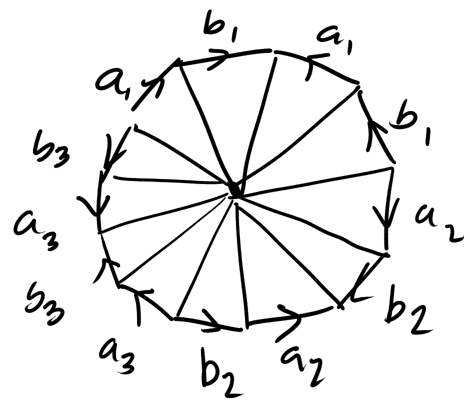
Add lots of vertices and edges to make triangulation that contains $T_1 \cup T_2$

This will be a refinement of each. $\boxtimes$

Consequence of lemma: $\chi(T_1) = \chi(T') = \chi(T_2)$ so we are done. $\boxtimes$

<u>Proposition:</u> the Euler characteristic of a genus g surface is $2-2g$.

<u>Proof:</u> Use any triangulation. For instance writing genus g surface as $4g$-gon with edges identified



2 vertices

$\frac{4g}{2}+4g=6g$ edges

$4g$ faces

$$\chi = 2-6g+4g = 2-2g$$

<u>Riemann-Hurwitz formula:</u> Let $F: X \to Y$ be a nonconstant map of compact, connected Riemann surfaces.

Then $$\chi(X) = \deg(F)\chi(Y) - \sum_{p\in X}\left[\mathrm{mult}_p(F)-1\right]$$

or $$2g(X)-2 = \deg(F)(2g(Y)-2) + \sum_{p\in X}\left[\mathrm{mult}_p(F)-1\right]$$

Note that since the set of points where $\mathrm{mult}_p(F)>1$ is finite, the sum in this formula is essentially finite.

<u>Proof:</u> Recall: $\mathrm{Ramification}(F) = \{p\in X \mid \mathrm{mult}_p(F)>1\}$

$\mathrm{Branch}(F) = F(\mathrm{Ramification}(F))$

Choose a triangulation T of Y such that every $y\in \mathrm{Branch}(F)$ is a vertex.

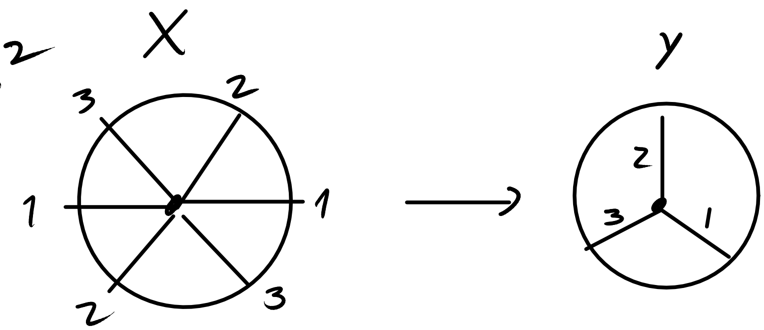
let $F^{-1}(T)$ be the triangulation on X given by taking preimages of all parts of T. Here we are using the fact that

$$F: X \setminus F^{-1}(\mathrm{Branch}(F)) \longrightarrow Y \setminus \mathrm{Branch}(F)$$

is a covering map.

Note that every ramification point is a vertex of $F^{-1}(T)$

$z \mapsto z^2$



let v, e, f be number of vertices, edges, faces of T
v', e', f' be " " " of $F^{-1}(T)$

Then $f' = \deg(F) f$ and $e' = \deg(F)\, e$

Vertices are different: if $y \in T$ is a vertex not at a branch point, $F^{-1}(T)$ has $\deg(F)$ vertices that map to y.

But if y is a branch point, there are fewer vertices in $F^{-1}(T)$ that map to y. only $\# F^{-1}(y)$

Recall $$\deg(F) = \sum_{p \in F^{-1}(y)} \mathrm{mult}_p(F)$$

so $$\sum_{p \in F^{-1}(y)} \left[\mathrm{mult}_p(F) - 1\right] = \deg(F) - \# F^{-1}(y)$$

or $$\# F^{-1}(y) = \deg(F) - \sum_{p \in F^{-1}(y)} \left[\mathrm{mult}_p(F) - 1\right]$$

So $$v' = \sum_{y \in \mathrm{Vert}(T)} \# F^{-1}(y) = \sum_{y \in \mathrm{Vert}(T)} \Big(\deg(F) - \sum_{p \in F^{-1}(y)} \big[\mathrm{mult}_p(F) - 1\big] \Big)$$

$$= \deg(F)\, v - \sum_{p \in X} \big[\mathrm{mult}_p(F) - 1\big]$$

Put it together

$$\chi(X) = v' - e' + f' = \deg(F)\, v - \sum_{p \in X} \big[\mathrm{mult}_p(X) - 1\big] - \deg(F)\, e + \deg(F)\, f$$

$$= \deg(F)\,(v - e + f) - \sum_{p \in X} \big[\mathrm{mult}_p(F) - 1\big]$$

$$= \deg(F)\, \chi(Y) - \sum_{p \in X} \big[\mathrm{mult}_p(F) - 1\big] \qquad \blacksquare$$

The quantity $\sum_{p \in X} \big[\mathrm{mult}_p(F) - 1\big]$ is called the total ramification of $F : X \to Y$.