Multiplicities and degrees

__Local normal form of a holomorphic map__ $F: X \to Y$ nonconstant

For each $p \in X$, there are local coordinates $\phi_1: U_1 \to V_1$ centered at $p \in X$ and $\phi_2: U_2 \to V_1$ centered at $F(p)$ such that in these coordinates,

$$\phi_2(F(\phi_1^{-1}(z))) = z^m.$$

For some integer $m \geq 1$.

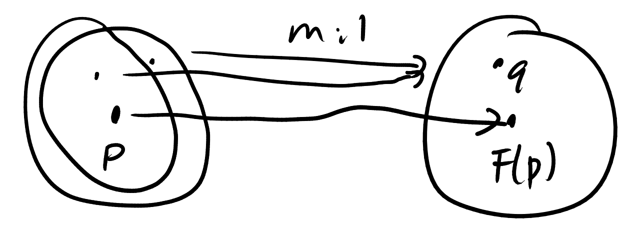
__Proof__: Let $f = \phi_2 \circ F \circ \phi_1^{-1}$ for any ϕ_1 centered at p, ϕ_2 centered at $F(p)$.
Then $f(z) = z^m g(z)$ for some $g(z)$ holomorphic $g(0) \neq 0$, $m = \mathrm{ord}_0(f)$.
Since $g(0) \neq 0$, we can construct an mth root $h(z)$ locally near $z = 0$. $h(z)^m = g(z)$. Thus $f(z) = (zh(z))^m$.

Now we can use $w = zh(z)$ as the local coordinate.
Indeed $\frac{dw}{dz} = zh'(z) + h(z)$ so $\frac{dw}{dz}(0) \neq 0$

Composing ϕ_1 with $z \mapsto w$ gives the desired chart. ∎

The value of m is uniquely determined by F and p. Indeed, it is the number of preimages near p of a point near $F(p)$.

__Def__: The multiplicity of F at p, $\mathrm{mult}_p(F)$, is the unique integer m s.t. F has the form $z \mapsto z^m$ at p



One can also calculate the multiplicity in any coordinate system.
Pick any coordinate $\phi_1 : U_1 \to V_1 \ni z$ near p
$\phi_2 : U_2 \to V_2 \ni w$ near $F(p)$
then F is expressed as $w = f(z)$ where $f = \phi_2 F \phi_1^{-1}$
let $z_0 = \phi_1(p)$. then

$$\mathrm{mult}_p(F) = 1 + \mathrm{ord}_{z_0}\left(\frac{df}{dz}\right)$$

Proof: The local normal form is given by

$$\underbrace{(f(z) - f(z_0))}_{\tilde{w}} = (\underbrace{(z - z_0)\, h(z)}_{\tilde{z}})^m$$

$$f(z) = f(z_0) + \sum_{i=m}^{\infty} c_i (z - z_0)^i$$

$$\frac{df}{dz} = \sum_{i=m}^{\infty} i c_i (z - z_0)^{i-1} \quad \text{so } \mathrm{ord}_{z_0} \frac{df}{dz} = m - 1 \quad \blacksquare$$

Note that $\mathrm{mult}_p(F) \geq 1$. For almost all points $p \in X$ $\mathrm{mult} = 1$, the points where this fails.

Def $F: X \to Y$ nonconstant, if $\mathrm{mult}_p(F) \geq 2$ then p is called a <u>ramification point</u>, and $F(p)$ is called a <u>branch point</u>.

(They are also the <u>critical points</u> and <u>critical values</u> of the map F)

Example: Let $X = \{(x,y) \mid f(x,y) = 0\} \subset \mathbb{C}^2$ be a smooth plane curve. $\pi: X \to \mathbb{C}$ defined by $\pi(x,y) = x$. Then p is a ramification point iff $\frac{\partial f}{\partial y}(p) = 0$.

Lemma: Let f be a meromorphic function on X, and let $F: X \to \hat{\mathbb{C}}$ denote corresponding map to Riemann sphere. Then

$$\mathrm{mult}_p(F) = \begin{cases} \mathrm{ord}_p(f - f(p)), & p \text{ not a pole} \\ -\mathrm{ord}_p(f), & p \text{ a pole.} \end{cases}$$

Proof: p not a pole: $\mathrm{mult}_p(F) = 1 + \mathrm{ord}_p \frac{df}{dz} = 1 + \mathrm{ord}_p \frac{d}{dz}(f - f(p))$
$= \mathrm{ord}_p\,(f - f(p))$.

p a pole: $\mathrm{mult}_p(F) = \mathrm{ord}_p(1/f) = -\mathrm{ord}_p(f)$.

Degree of a holomorphic map for compact RS.

Proposition: $F: X \to Y$ nonconstant. X, Y compact connected. For $y \in Y$ define

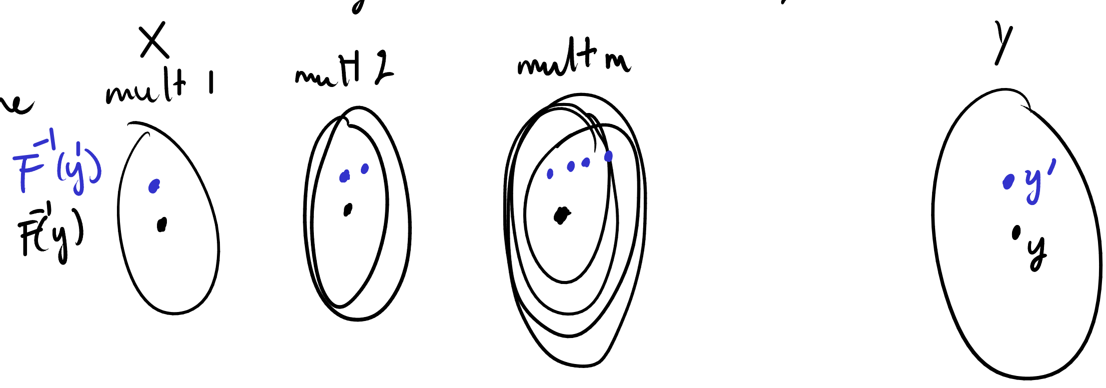
$$d_y(F) = \sum_{x \in F^{-1}(y)} \mathrm{mult}_x(F) \qquad (\text{degree at } y \in Y)$$

Then $d_y(F)$ is independent of y.

Def $d(F) = d_y(F)$ is called the degree of F, $\deg(F)$

Proof of prop: We claim that $d_y(F)$ is locally constant in y. meaning that $d_y(F) = d \Rightarrow d_{y'}(F) = d$ for all y' near y. It will follow that $d_y(F)$ is constant since Y is connected.

Local picture



For y' near y, there are exactly $d_y(F)$ preimages, and multiplicity of F is one at each of these points. thus

$$d_y(F) = d_{y'}(F) \quad \blacksquare$$

Observation: $\deg(F) = 1 \iff F$ is an isomorphism

<u>Corollary</u>: X compact RS, f a meromorphic function with a single simple pole. Then the corresponding $F: X \to \hat{\mathbb{C}}$ is an isomorphism.

<u>Proof</u>: p pole: $\deg(F) = d_\infty(F) = \mathrm{mult}_p(F) = -\mathrm{ord}_p(f) = -(-1) = 1$.

<u>Prop</u> let f be nonconstant meromorphic function on a compact RS X. Then

$$\sum_{p \in X} \mathrm{ord}_p(f) = 0$$

<u>proof</u> let $x_1, \dots, x_r$ be zeros of f, $y_1, \dots, y_s$ poles of f. let $F: X \to \hat{\mathbb{C}}$ be corresponding map.

Then $\{x_1, \dots, x_r\} = F^{-1}(0)$

$\{y_1, \dots, y_s\} = F^{-1}(\infty)$

By constancy of degree:

$$\deg(F) = d_0(F) = \sum_{i=1}^{r} \mathrm{mult}_{x_i}(F)$$

$$\deg(F) = d_\infty(F) = \sum_{j=1}^{s} \mathrm{mult}_{y_j}(F)$$

Now $\mathrm{mult}_{x_i}(F) = \mathrm{ord}_{x_i}(f)$, $\mathrm{mult}_{y_j}(F) = -\mathrm{ord}_{y_j}(f)$

Thus $\sum_i \mathrm{ord}_{x_i}(f) = -\sum_j \mathrm{ord}_{y_j}(f)$, $\sum_{i=1}^{r} \mathrm{ord}_{x_i}(f) + \sum_{j=1}^{s} \mathrm{ord}_{y_j}(f) = 0 \quad \blacksquare$