

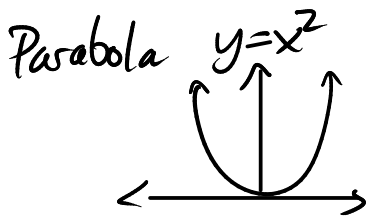
# Riemann Surfaces and Algebraic Curves

Riemann Surfaces are 2d topological manifolds endowed with an extra structure, roughly, a notion of local complex coordinates.

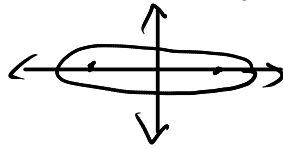
⇒ We can talk about holomorphic/meromorphic functions when the domain is a Riemann surface.

⇒ can generalize complex analysis.

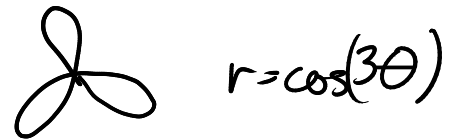
Algebraic curves (in the plane, say) Originally, these were just curves defined by algebraic equations.



Ellipse  $ax^2 + by^2 = 0$

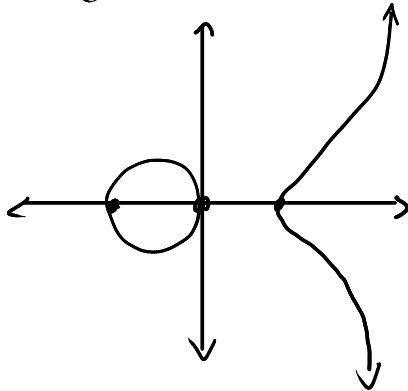


3-petal flower



(yes, this is algebraic)

$$y^2 = x(x-1)(x+1)$$



All kinds of interesting/pretty shapes.

There is a fairly straightforward way to produce a Riemann surface from an equation for an algebraic curve.

Given eqn  $F(x,y)=0$ , let  $S = \{(x,y) \in \mathbb{C}^2 \mid F(x,y)=0\}$  be the set of complex solutions.

Under certain conditions (nonsingularity of  $F$ ),  $S$  is a Riemann surface.

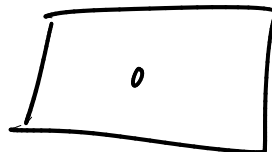
Eg.

$$F(x,y) = x, \quad S = \{(x,y) \in \mathbb{C}^2 \mid x = 0\}$$

$y$  is a natural coordinate on  $S$ , and  $S = \text{plane}$

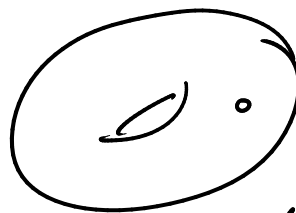
$$F(x,y) = x^2 + y^2, \quad S = \{(x,y) \in \mathbb{C}^2 \mid x^2 + y^2 = 0\}$$

In fact  $S = \text{plane minus a point}$



$$F(x,y) = y^2 - x(x-1)(x+1), \quad S = \{(x,y) \in \mathbb{C}^2 \mid y^2 - x(x-1)(x+1) = 0\}$$

In fact  $S = \text{Torus minus a point}$



Being able to translate between the two sides is a theme of the course. In fact there is an equivalence between

$$\{\text{compact Riemann surfaces}\} \longleftrightarrow \{\text{smooth complete algebraic curves over } \mathbb{C}\}$$

The adjectives here will require some explanation, but that will come in due time.

Riemann surfaces can be studied with complex analysis, whereas algebraic curves can be studied with abstract algebra (such as field theory). This course will start with the analytic perspective, and gradually transition to a more algebraic one.

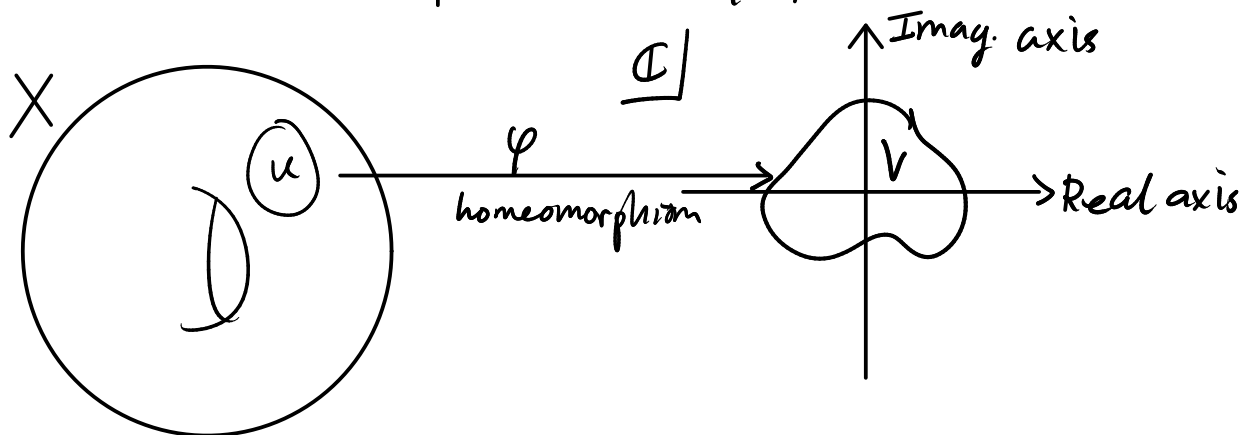
## Basic definitions:

Complex charts: let  $X$  be a topological space.

Def A complex chart on  $X$  is a function  $\varphi: U \rightarrow V$  where

- $U \subset X$  is an open set
- $V \subset \mathbb{C}$  is an open set
- $\varphi: U \rightarrow V$  is a homeomorphism.

$U$  is called the domain of  $\varphi$ , and  $\varphi$  is said to be centered at  $p \in U$  if  $\varphi(p) = 0$



Thus  $\varphi$  is a complex-valued function on  $U$ . We think of the value of  $\varphi$  as a complex coordinate in  $U$ .

We sometimes use the notation  $z = \varphi(x)$   $\begin{pmatrix} x \in U \\ z \in \mathbb{C} \end{pmatrix}$

Def

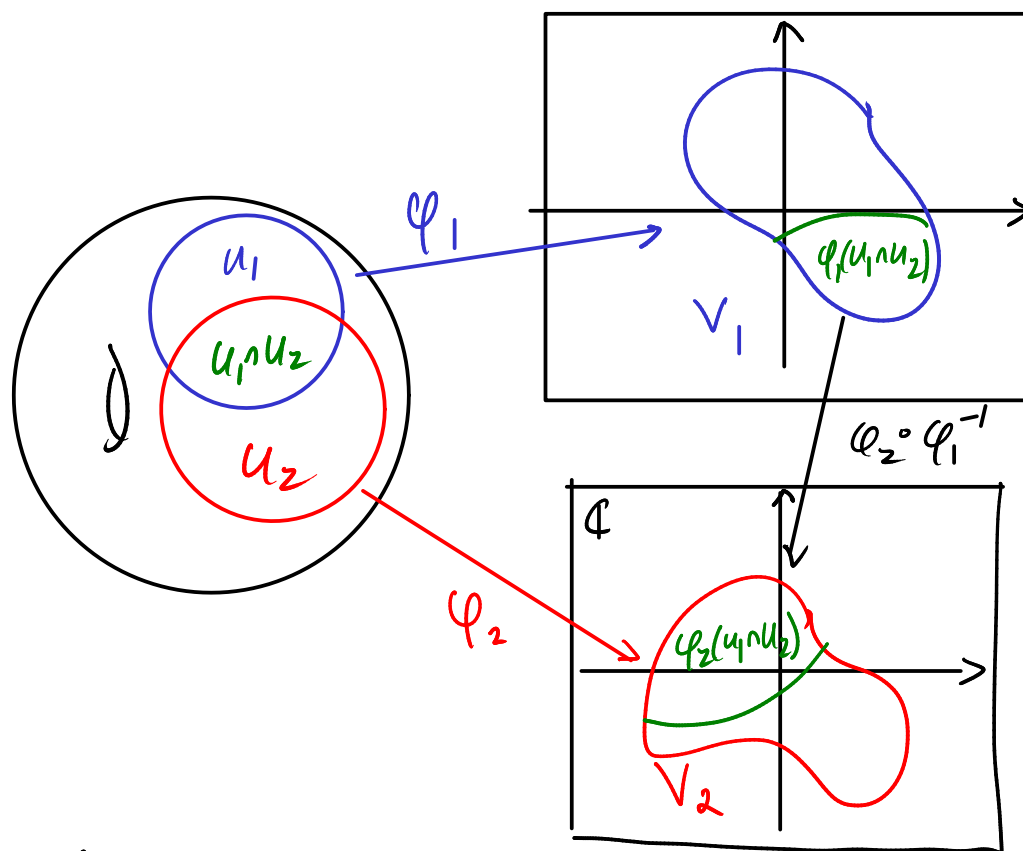
Two charts  $\varphi_1: U_1 \rightarrow V_1$  and  $\varphi_2: U_2 \rightarrow V_2$  are compatible if either  $U_1 \cap U_2 = \emptyset$

or

$$\varphi_2 \circ \varphi_1^{-1}: \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$$

is holomorphic.

Picture



$\varphi_2 \circ \varphi_1^{-1}$  is a map between open subsets of  $\mathbb{C}$ , and it is automatically a homeomorphism.

As  $\varphi_2 \circ \varphi_1^{-1}$  is a complex-valued function on an open subset in  $\mathbb{C}$ , it makes sense to ask for it to be holomorphic

[Recall: A function  $f: V \rightarrow \mathbb{C}$  defined on open set  $V \subset \mathbb{C}$  is holomorphic if, for each  $z_0 \in V$ ,

$$f'(z_0) := \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists as a complex number.}$$

This condition is equivalent to the Cauchy-Riemann equations  
 $z = x + iy$ ,  $f(x + iy) = u(x, y) + iv(x, y)$

$$(CR) \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

And to the condition that  $f$  has convergent power series in terms of  $z$ :

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Examples: let  $X = \mathbb{C} \cup \{\infty\}$  be the one point compactification of the complex plane. The topology of  $X$  is defined by declaring a set  $U \subset X$  to be open if either

- (a)  $\infty \notin U$ , and  $U \subset \mathbb{C}$  is open
- (b)  $\infty \in U$  and  $X \setminus U$  is a compact subset of  $\mathbb{C}$

let  $U_1 = \mathbb{C} \subset X$  and  $\varphi_1: U_1 \rightarrow \mathbb{C}$  the identity map.

let  $U_2 = (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$  and  $\varphi_2: U_2 \rightarrow \mathbb{C}$  the map

$$\varphi_2(z) = \begin{cases} 1/z & z \neq \infty \\ 0 & z = \infty \end{cases}$$

One must check that  $\varphi_2$  is a homeomorphism of  $U_2$  to  $\mathbb{C} \dots$

These charts are compatible:  $U_1 \cap U_2 = \mathbb{C} \setminus \{0\}$

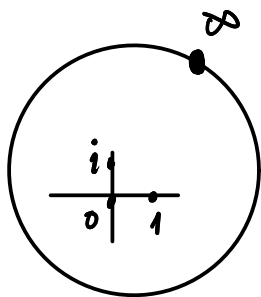
$$\varphi_1(U_1 \cap U_2) = \mathbb{C} \setminus \{0\} \quad \varphi_2(U_1 \cap U_2) = \mathbb{C} \setminus \{0\}$$

and  $\varphi_2 \circ \varphi_1^{-1}: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$  is the function  $z \mapsto \frac{1}{z}$

which is indeed holomorphic on the domain  $\mathbb{C} \setminus \{0\}$ .

So  $\varphi_1$  and  $\varphi_2$  are compatible.

$X = \mathbb{C} \cup \{\infty\}$  is a Riemann surface called the Riemann sphere.



Nonexample:  $X = \mathbb{C}$

$$\begin{array}{ll} U_1 = \mathbb{C} & U_2 = \mathbb{C} \\ \varphi_1: \mathbb{C} \rightarrow \mathbb{C} & \varphi_2: \mathbb{C} \rightarrow \mathbb{C} \\ \varphi_1(z) = z & \varphi_2(z) = \frac{z}{1+|z|^2} \end{array}$$

$(\varphi_2 \circ \varphi_1^{-1})(z) = \frac{z}{1+|z|^2}$  not holomorphic!  $\Rightarrow$  Not compatible