RIEMANN SURFACES MIDTERM EXAM: DUE MONDAY MARCH 14

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For this exam, you may consult the textbook, the lecture notes, your own notes, and your previous homeworks, but not other references. You should not discuss the exam with any one but me (JP).

(1) Let X be a Riemann surface. Denote by $\mathcal{U}$ the set of open sets of X. For each open set $U \in \mathcal{U}$, define

$$\mathcal{O}_X(U) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\} \subset \mathbb{C}^U,$$

that is, $\mathcal{O}_X(U)$ is the set of complex-valued functions on U that are holomorphic (with respect to the given complex structure of X). In sum, we can associate to the Riemann surface X the collection $\{\mathcal{O}_X(U)\}_{U\in\mathcal{U}}$ of sets of functions on the various open sets $U \in \mathcal{U}$.

(a) Show that the data $\{\mathcal{O}_X(U)\}_{U\in\mathcal{U}}$ determine the complex structure of X. That is, show that if the topological space $(X, \mathcal{U})$ is given, and for each open set $U \in \mathcal{U}$, a set of functions $A(U) \subset \mathbb{C}^U$ is given, then there is *at most one* complex structure on X such that $\mathcal{O}_X(U) = A(U)$.

(b) Let X and Y be Riemann surfaces. Denote by $\mathcal{U}_X$ and $\mathcal{U}_Y$ the associated sets of open sets. Let $F : X \to Y$ be a continuous map. Show that F is holomorphic if and only if

$$(\forall V \in \mathcal{U}_Y)(\forall f : V \to \mathbb{C})(f \in \mathcal{O}_Y(V) \implies f \circ F \in \mathcal{O}_X(F^{-1}(V))),$$

in other words, whenever we have a holomorphic function $f : V \to \mathbb{C}$ on an open set $V \subset Y$, the pull-back $f \circ F : F^{-1}(V) \to \mathbb{C}$ is a holomorphic function on $F^{-1}(V) \subset X$.

Hint for both parts: think of the charts $\phi : U \to \mathbb{C}$ on X as elements of $\mathcal{O}_X(U)$.

(2) Let $X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = \sin x\}$. Even though the defining function $y^2 - \sin x$ is not a polynomial, use the implicit function theorem to show that X has the structure of a Riemann surface. Find all ramification points of the map $\pi : X \to \mathbb{C}$, $\pi(x, y) = x$. Give an intuitive argument why X cannot be realized as a open subset of a compact Riemann surface.

(3) In $\mathbb{CP}^2$ with homogeneous coordinates $[x : y : z]$, and for each positive integer d, define a curve X_d by the equation $x^d + y^d + z^d = 0$ (the *Fermat curve*). There is a map $F : X_d \to X_1$ given by raising the coordinates to the power d: $F([x : y : z]) = [x^d : y^d : z^d]$.

   (a) Verify that X_d is a smooth curve, and that F is a holomorphic map.
   (b) Determine the degree of F, as well as its ramification points and their multiplicities.
   (c) Using the information from the previous part, determine the genus of X_d.

(4) Let L be a lattice in $\mathbb{C}$, and let $X = \mathbb{C}/L$ be the corresponding complex torus. Show that the map $\sigma : \mathbb{C} \to \mathbb{C}$, $\sigma(z) = -z$ descends to a well-defined holomorphic map $\bar{\sigma} : X \to X$, and that applying $\bar{\sigma}$ twice gives the identity. Describe the quotient $Y = X/\bar{\sigma}$, and the quotient map $\pi : X \to Y$. Your description should include
   (a) the genus of Y,
   (b) the degree of π,
   (c) the ramification points of π, and their multiplicities.

(5) Let X be the affine plane curve defined by the equation $y^2 = x^3 - x^2$.
   (a) Find the singular points of X, describe them and their resolutions.
   (b) Show that X has a single hole at infinity. (More precisely, there is a hole chart on the smooth part of X whose domain is the complement of a compact subset of X.)
   (c) Find the genus of the surface obtained by resolving the singular points and filling the hole at infinity.

(6) Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces X and Y. Suppose that X has genus 4, Y has genus 2, and that F has at least one ramification point. Show that the Riemann-Hurwitz formula and other general facts about holomorphic maps determine the degree of F and the number of ramification points of F.