RIEMANN SURFACES FINAL EXAM

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- For this exam, you may consult your textbook and notes.
- Unless otherwise specified, all Riemann surfaces are assumed to be compact and connected.
- Please complete four (4) of the following problems.

(1) Consider the following situation:
  (a) G is a finite group,
  (b) G acts effectively and holomorphically on a Riemann surface X,
  (c) $p \in X$ is a point with trivial stabilizer: $G_p = \{e\}$,
  (d) $g \in G$ is a central element (that is $gh = hg$ for all $h \in G$), and
  (e) $\gamma : [0, 1] \to X$ is a path with $\gamma(0) = p$, $\gamma(1) = g.p$, and the image of γ is disjoint from the set of points in X with nontrivial stabilizer.

  Denote by $\pi : X \to X/G$ the projection. Show that:
  (a) $\pi \circ \gamma : [0, 1] \to X/G$ is a loop whose image is disjoint from the set of branch points of π.
  (b) The monodromy of the branched covering $\pi : X \to X/G$ along the loop $\pi \circ \gamma$ coincides with the permutation of $\pi^{-1}(\pi(p))$ induced by the action of the given element $g \in G$.

(2) Let X be the smooth affine plane curve defined by a polynomial $f(x, y)$:
$$X = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$$
  (Thus X is not compact.) Show that $p(x, y)dx + q(x, y)dy$ defines a holomorphic 1-form on X if $p(x, y)$ and $q(x, y)$ are polynomials. Show that
$$\frac{\partial f}{\partial x} dx = -\frac{\partial f}{\partial y} dy$$
  when both sides of this equation are interpreted as 1-forms on X.

(3) The *gonality* γ of X is defined to be the minimal degree of a non-constant holomorphic map $F : X \to \mathbb{P}^1$
$$\gamma := \min\{\deg F \mid F : X \to \mathbb{P}^1 \text{ nonconstant}\}.$$
  (a) Show that γ is also equal to the minimal degree of a divisor D such that $\dim L(D) \geq 2$:
$$\gamma = \min\{\deg D \mid D \in \mathrm{Div}(X), \dim L(D) \geq 2\}.$$
  (b) Suppose that X has genus 3. Show that the gonality of X is either 2 or 3.

(4) Let X be a Riemann surface of genus g. Let D be a divisor of degree $2g+1$. Recall that D defines an embedding $\phi_D : X \to \mathbb{P}^r$, where $r = \ell(D) - 1$. Show that there is a homogeneous polynomial $F(x_0, \ldots, x_r)$ that vanishes on the image $\phi_D(X)$. Prove an upper bound for the minimal degree of this polynomial (your bound doesn't need to be sharp.)

(5) Let X be a genus 2 Riemann surface. There is an Abel-Jacobi map

$$A_2 : \mathrm{Sym}^2(X) \to \mathrm{Jac}(X)$$

from the symmetric product $\mathrm{Sym}^2(X) = (X \times X)/S_2$ to the Jacobian. Show that the fibers of this map are either points $\mathbb{P}^0$ or projective lines $\mathbb{P}^1$, and that in fact, all fibers but one are points.