Numerical approximation: Euler's method.

Some times, one may encounter a differential equation that cannot be solved symbolically: e.g.

$$\frac{dy}{dx} = e^{-x^2}$$

solution $y = \int e^{-x^2} dx$ is not an "elementary" function.

How can we calculate such a solution, at least approximately?

Initial value problem

$$\begin{cases} \frac{dy}{dx} = f(x, y(x)) & \leftarrow \text{tells us how } y \text{ changes} \\ y(x_0) = y_0 & \leftarrow \text{tells us where } y \text{ starts.} \end{cases}$$

Euler's method:

- Start at point (x_0, y_0) as specified by initial condition.
- Use Diff. eq. to estimate how y will change over a short interval: At (x_0, y_0), slope is $f(x_0, y_0)$.
  Over the interval x_0 to x_0+h, y will change by approximately $\Delta y = h \cdot f(x_0, y_0)$, so y will be approximately
  $y_1 = y_0 + h \cdot f(x_0, y_0)$.

[This <u>would</u> be exact <u>if</u> $\frac{dy}{dx}$ were constant on $[x_0, x_0+h]$]

Then repeat starting with $x_1 = x_0+h$ and $y_1 = y_0 + h \cdot f(x_0, y_0)$

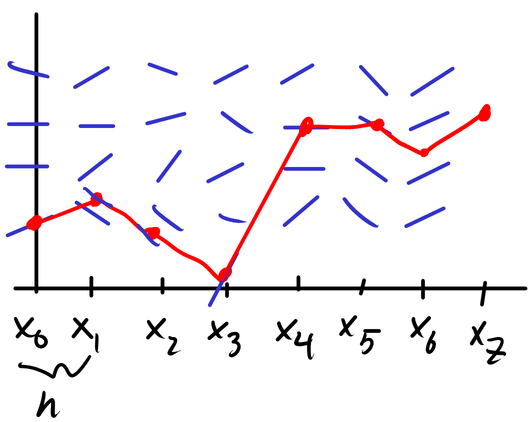
Euler iteration: x_0 $\quad y_0$ come from initial condition

$$x_1 = x_0 + h \qquad y_1 = y_0 + h\cdot f(x_0, y_0)$$
$$x_2 = x_1 + h \qquad y_2 = y_1 + h\cdot f(x_1, y_1)$$
$$x_3 = x_2 + h \qquad y_3 = y_2 + h\cdot f(x_2, y_2)$$
$$\vdots$$
$$x_{n+1} = x_n + h \qquad y_{n+1} = y_n + h\cdot f(x_n, y_n)$$

Compute enough steps to reach any desired x-value.

h is a parameter in the method, called the <u>step size</u>.

Graphically: Go where the slope field tells you.



This method is computationally intensive, especially with small step size

<u>Example</u>: Use Euler's method to approximate $y(10)$, where $y(x)$ is the solution of $\left\{ \frac{dy}{dx} = x + y, \quad y(0) = 1 \right\}$

What step size? How about $h = 1$

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	1	2	3	4	5	6	7	8	9	10
y_n	1	2	5	12	27	58	121	248	503	1014	$\boxed{2037}$
$h\cdot f(x_n, y_n) = x_n + y_n$	1	3	7	15	31	63	127	255	511	1023	

$y(10) \approx 2037$

Or approximate $y(1)$ using $h = .2 = 1/5$

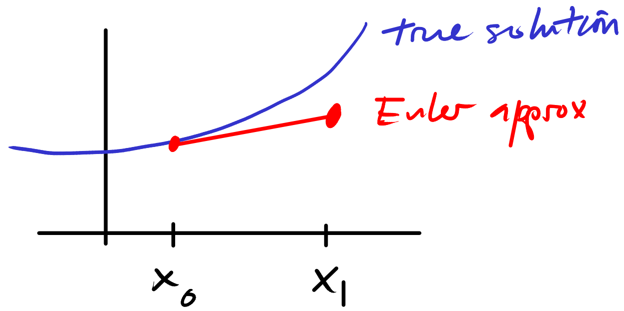
n	0	1	2	3	4	5
x_n	0	.2	.4	.6	.8	1.0
y_n	1	1.2	1.48	1.856	2.347	$\boxed{2.976}$
$h \cdot f(x_n, y_n)$	.2	.28	.376	.491 (rounding here)	.629	

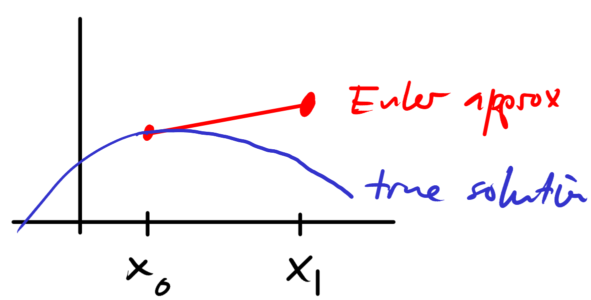
$y(1) \approx 2.976$

[vs. $y(1) \approx 3$ from before]

different step sizes will give different answers.
Smaller step size and more steps leads to better approximations.
[Though, extremely small step size leads to challenges with representing high precision arithmetic on the computer.]

Error analysis: Like any approximate method, Euler's method introduces errors.





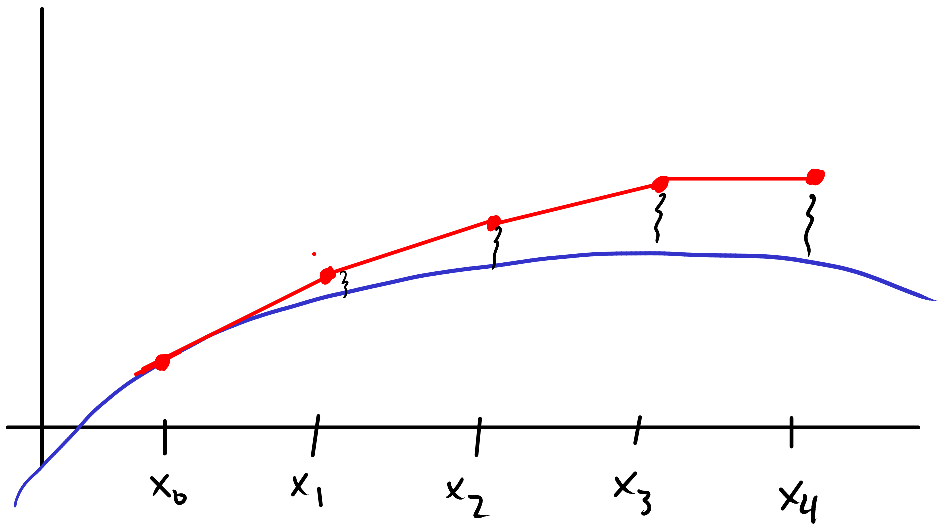
True solution concave up
$\Rightarrow$ Euler approximation is an underestimate

True solution concave down
$\Rightarrow$ Euler approximation is an over estimate.

More sophisticated numerical methods exist (e.g. Runge-Kutta) that do better with this.

Another issue is that errors may add up (accumulate)

Eg.



Each step introduces some error, and they may all have the same sign.

Also the error propagates from one step to the next

$$y_{n+1} = y_n + h \cdot f(\underbrace{x_n, y_n})$$

this point is not actually on the solution curve we are trying to find, so the true slope could be slightly different.

Lastly, there is the issue of rounding / adequate representation of decimal numbers.

In spite of these limitations, Euler's method is fundamental to numerical solution of differential equations.