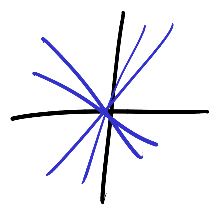
Existence and uniqueness ; Picard iteration

Suppose we have a differential equation initial value problem :
$$\begin{cases} \dfrac{dy}{dx} = f(x,y(x)) \\ y(a) = b \end{cases}$$

How do we know it has any solutions at all?
Will the solutions be unique?

Recall
$$\left\{ \begin{array}{l} \dfrac{dy}{dx} = \dfrac{y}{x} \\ y(0) = b \end{array} \right\}$$



Has no solution if $b \neq 0$
infinitely many if $b = 0$

Or: what if equation is really wierd and we can't find solutions no matter how hard we try?

Goal: Build a solution "abstractly".
An algorithm that applies to any first order ordinary differential equation: <u>Picard iteration.</u>

Suppose y satisfies
$$\begin{cases} \dfrac{dy}{dx} = f(x,y(x)) \\ y(a) = b \end{cases}$$

Integrate $\int_a^c dx$:
$$\int_a^c \frac{dy}{dx}\,dx = \int_a^c f(x,y(x))\,dx$$
$$\|$$
$$y(c) - y(a) = y(c) - b$$

so $y(c) = b + \int_a^c \underbrace{f(x, y(x))\,dx}$ for every c.

x is a dummy variable
rename $x \to t$
$c \to x$

equivalent to write
$$y(x) = b + \int_a^x f(t, y(t))\,dt$$

In fact: $\left\{ \begin{array}{l} \frac{dy}{dx} = f(x, y(x)) \\ y(a) = b \end{array} \right\}$ is equivalent to this equation

Consider the operator:

function $y(x) \longrightarrow$ function $P[y](x) = b + \int_a^x f(t, y(t))\,dt$

$y(x)$ solves the IVP if and only if $P[y](x) = y(x)$.

To produce a solution:

1) start with constant function $y_0(x) = b$
2) apply P many times $y_1 = P[y_0]$, $y_2 = P[y_1]$, $y_{n+1} = P[y_n]$
3) The functions $y_n(x)$ are <u>approximate solutions</u>

of $y(x) = \lim_{n\to\infty} y_n(x)$. In <u>good cases</u>, this will be a <u>genuine solution</u>.

Sketch: let $y_0(x) = b$

let $y_{n+1}(x) = \underline{P}[y_n](x) = b + \int_a^x f(t, y_n(t))\,dt$

Let $y(x) = \lim_{n\to\infty} y_n(x)$ (Have to justify this exists)

Now what is $P[y]$?

$$P[y](x) = b + \int_a^x f(t, y(t))\,dt = b + \int_a^x f(t, \lim_{n\to\infty} y_n(t))\,dt$$

needs justification

$$= \lim_{n\to\infty}\left[b + \int_a^x f(t, y_n(t))\,dt\right] = \lim_{n\to\infty} y_{n+1}(x) = y(x)$$

so $\underline{P}[y] = y$, and y is a genuine solution of $\left\{\begin{array}{l}\frac{dy}{dx} = f(x, y(x)) \\ y(a) = b\end{array}\right\}$!

Example: $\left\{\begin{array}{l}\frac{dy}{dx} = -y \\ y(0) = 1\end{array}\right\}$ $\quad f(x,y) = -y$

$y_0(x) = 1$

$$y_1(x) = P[y_0](x) = 1 + \int_0^x (-y_0(t))\,dt = 1 + \int_0^x (-1)\,dt$$

$$= 1 - x$$

$$y_2(x) = 1 + \int_0^x - y_1(t)\,dt = 1 + \int_0^x -(1-t)\,dt = 1 - x + \frac{x^2}{2}$$

$$y_3(x) = 1 + \int_0^x -\left(1 - t + \frac{t^2}{2}\right)dt = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

$$y_4(x) = 1 + \int_0^x -\left(1 - t + \frac{t^2}{2} - \frac{t^3}{6}\right)dt = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$$

We see a pattern: $y_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!}$

so $y(x) = \lim_{n\to\infty} y_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = e^{-x}$

And indeed, we could have gotten $y = e^{-x}$ as the solution by other methods.

Under what conditions does this actually work?
Depends on $f(x,y)$

Suppose $f(x,y)$ is continuous as a function of two variables
And there is a constant C such that

$$|f(x,y_1) - f(x,y_2)| \leq C\,|y_1 - y_2|$$

for all values of x, y_1, y_2

Then the Picard iteration $y = \lim_{n\to\infty} y_n$ always converges to a solution of the IVP $\begin{cases} \frac{dy}{dx} = f(x, y(x)) \\ y(a) = b \end{cases}$

Furthermore $y(x)$ is the unique solution of this IVP.