Autonomous equations

t independent variable
y dependent variable

$$\frac{dy}{dt} = f(y)$$

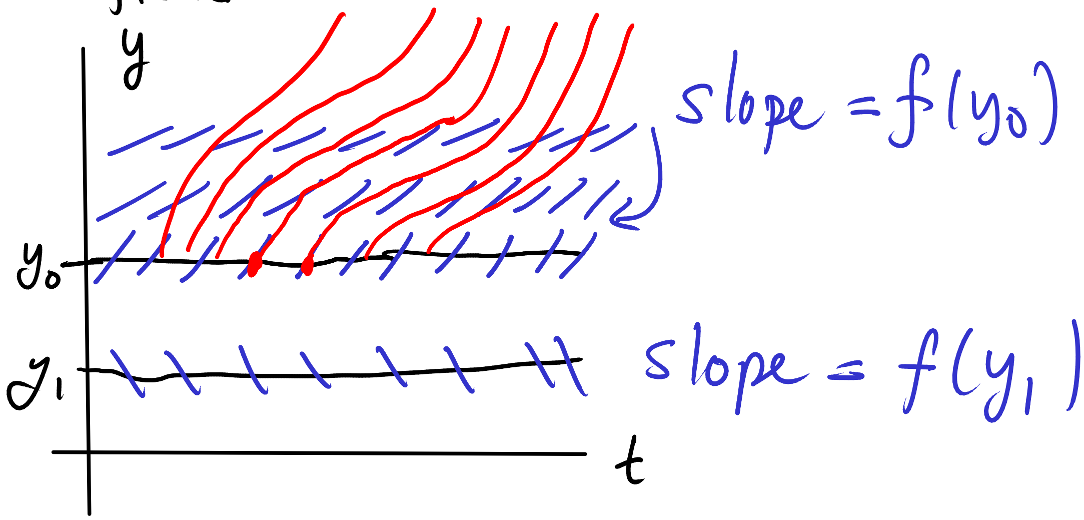
Note: There is no t on the RHS.

This is called autonomous equation.
(aka time-independent / time-invariant)

- Could solve this by separation of variables
- Can get a lot of qualitative info without explicitly solving the equation.

$$\frac{dy}{dt} = f(y)$$

Slope field



- slope is constant along horizontal lines

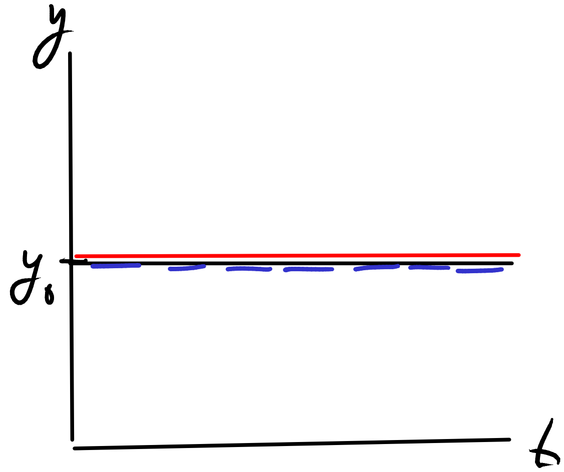
- if we have one solution curve, shifting it to the left or right by any amount yields another solution curve.

- If we shift time axis, the equation doesn't change.

- If $y(t)$ is a solution then so is $y(t+c)$.

Understanding solutions of $\frac{dy}{dt} = f(y)$

Suppose that y_0 is such that $f(y_0) = 0$



/ = slope field

— = solution curve

$$y(t) = y_0 \ (\text{constant})$$

<u>Definition</u>: a point y_0 is called a critical point (aka equilibrium point) if any of the following equivalent conditions hold:

- $f(y_0) = 0$
- slope at $y_0 = 0$
- $y(t) = y_0$ is a constant solution of the ODE

<u>Proof</u> that $f(y_0) = 0 \Rightarrow y(t) = y_0$ is solution.

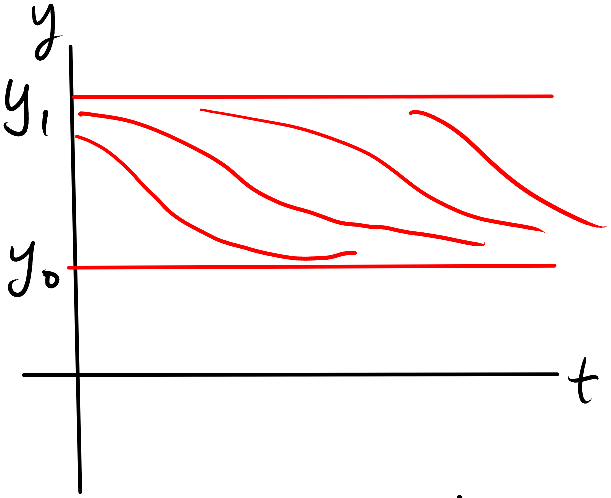
$$\frac{d}{dt}[y(t)] \quad \text{vs} \quad f(y(t))$$

$$\frac{d}{dt}[y(t)] = \frac{d}{dt}(y_0) = 0$$

$$f(y(t)) = f(y_0) = 0$$

Suppose we have two critical points y_0, y_1

in this region, $f(y) < 0$



other solutions can not cross the critical points (constant solutions)

Step 1: find critical points $f(y_0) = 0$

Step 2: what happens between critical points?

$\Rightarrow$ Draw the graph of $f(y)$

$\rightarrow$ figure out where $f(y)$ is $\begin{cases} \text{positive.} \\ \text{negative.} \end{cases}$

$\frac{dy}{dt} = f(y)$: if $f(y)$ is positive, then y is increasing

if $f(y)$ is negative, then y is decreasing.

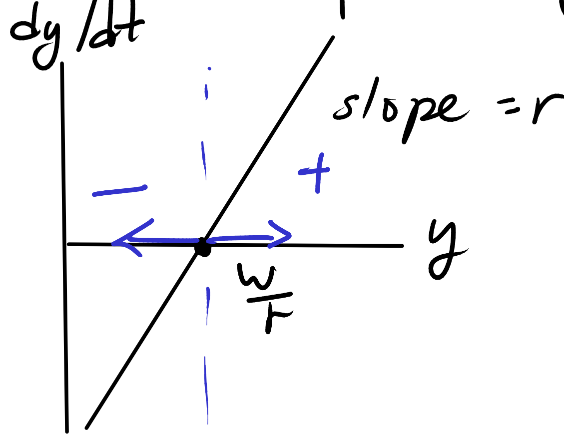
<u>Example</u> y = money in investment account

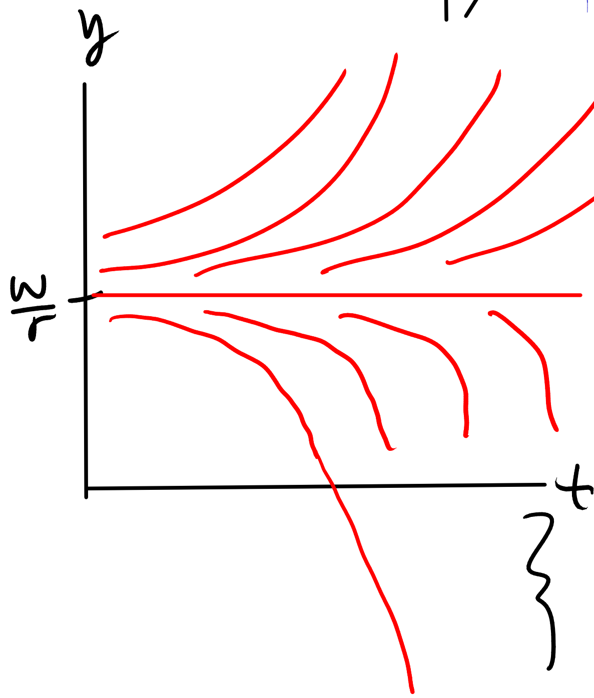
r = interest rate (continuously compounded)

$$\frac{dy}{dt} = ry - w$$ fees

$\frac{dy}{dt} = ry - W$ critical point $y_0 = \frac{W}{r}$

Graph $f(y)$





A solution bigger than $\frac{w}{r}$ increases

A sol^{n} less than w/r decreases.

no physical meaning.

Logistic equation (populations) $y = \#$ in pop.

Natural growth $\frac{dy}{dt} = ky$

→ k is constant   exponential growth

→ logistic growth, k is not constant

as y increases, k should decrease
due limitations on food, shelter, etc

simplest possibility $k = a - by$

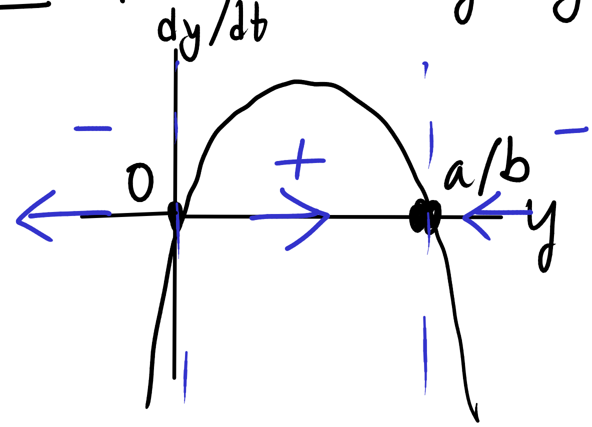
(linear $a, b > 0$)

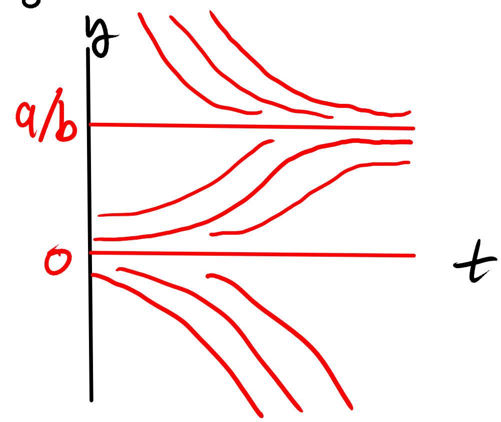
$$\frac{dy}{dt} = (a-by)\cdot y = ay - by^2 \quad \text{(Logistic equation)}$$

Step 1 critical points $(a-by_0)\cdot y_0 = 0$

$$\Rightarrow y_0 = \begin{cases} 0 \\ a/b \end{cases}$$

<u>Step 2</u> plot $(a-by)\cdot y$ | Solutions





Any solution where $y > 0$ asymptotically stabilizes at a/b

a/b is called the "carrying capacity" in population examples.

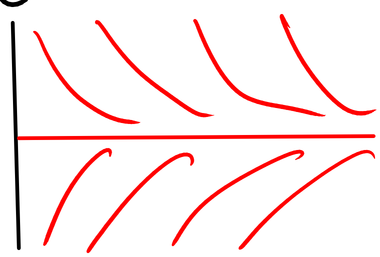
a/b is "attractive" stable critical point

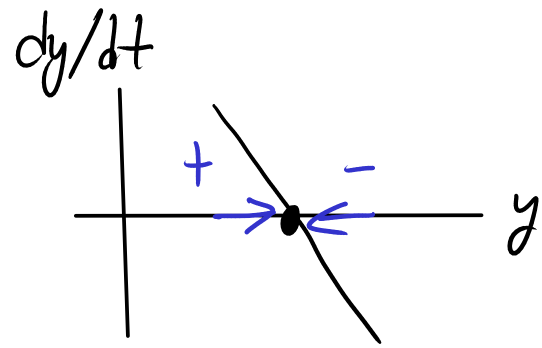
Stable equilibrium

0 is "repulsive" unstable critical point

unstable equilibrium.

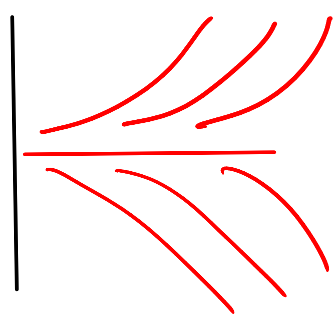
Stable

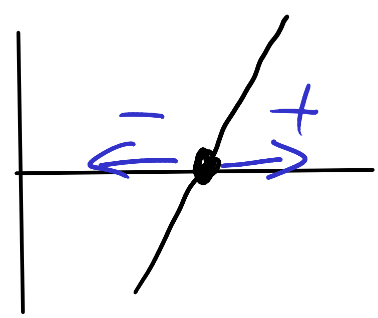




[restoring force or negative feed back loop]

Unstable



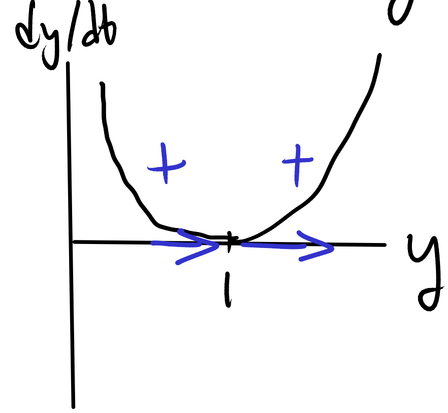


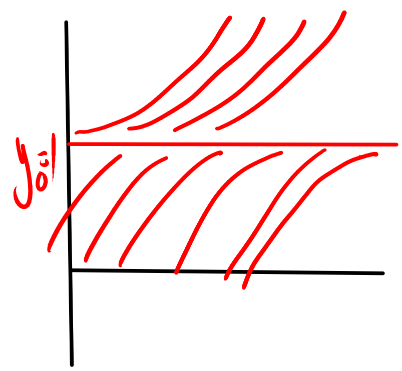
[amplifying force or positive feedback loop]

One more possibility:

$f(y) = (y-1)^2$

$y_0 = 1$ is the only critical point





Neither stable nor unstable: semistable.