First Order Linear equations (integrating factors)

This is an important and rather general class of equations that can be solved in terms of integrals

Definition a first-order linear equation is one that can be written as

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$P(x)$ and $Q(x)$ are the coefficient functions.

Call Linear because $\frac{dy}{dx}$ and y appear to first power ONLY.

Eg $\frac{dy}{dx} = xy \iff \frac{dy}{dx} - xy = 0$ $\quad P(x) = -x$, $Q(x) = 0$

$\frac{dT}{dt} = -k(T-A) \iff \frac{dT}{dt} + kT = kA$ $\quad P(t) = k$, $Q(t) = kA$

$x\frac{dy}{dx} - y = x^3 \iff \frac{dy}{dx} - \frac{1}{x}y = x^2$ $\quad P(x) = -\frac{1}{x}$, $Q(x) = x^2$

* We always want the equation in STANDARD FORM

$$\frac{dy}{dx} + P(x)y = Q(x)$$

To solve, we want to be able to integrate LHS, meaning we need to <u>recognize it as a derivative</u>.

The trick is to multiply the <u>whole equation</u> by a function $\rho(x)$ first.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\underbrace{\rho(x)\frac{dy}{dx} + \rho(x)P(x)y}_{} = \rho(x)\,Q(x)$$

This is reminiscent of the product rule

$$\frac{d}{dx}\left(\rho(x)y\right) = \rho(x)\frac{dy}{dx} + \frac{d\rho}{dx}y$$

What if we pick $\rho(x)$ so that $\frac{d\rho}{dx} = \rho(x)\,P(x)$?

This equation is separable:

$$\frac{1}{\rho}\frac{d\rho}{dx} = P(x) \longrightarrow \int \frac{1}{\rho}\,d\rho = \int P(x)\,dx$$

so $\ln|\rho| = \int P(x)\,dx$

$$\rho = Ce^{\int P(x)\,dx}$$

We only need one such ρ, so we just take

$$\rho(x) = e^{\int P(x)\,dx}$$

[This happens to be a situation where the constant of integration doesn't matter.]

Going back to the equation, with $\rho(x) = e^{\int P(x)dx}$

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x)$$

$$\underbrace{\phantom{e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y}}_{\parallel}$$

$$\frac{d}{dx}\left(e^{\int P(x)dx}y\right) = e^{\int P(x)dx}Q(x)$$

Can finally integrate!

$$e^{\int P(x)dx}y = \int\left(e^{\int P(x)dx}Q(x)\right)dx + C$$

Now the constant does matter.

Solve for y:

$$y = e^{-\int P(x)dx}\int\left(e^{\int P(x)dx}Q(x)\right)dx + Ce^{-\int P(x)dx}$$

Let's see it in action

$$\frac{dy}{dx} + 3y = 2xe^{-3x} \qquad \begin{aligned} P(x) &= 3 \\ Q(x) &= 2xe^{-3x}\end{aligned}$$

Step 1 find integrating factor $e^{\int P(x)dx}$ $\quad \int 3dx = 3x$

So we can use $\boxed{e^{3x}}$

Step 2: multiply the entire equation by the Integrating factor

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{3x}\cdot 2\cdot x\cdot e^{-3x} = 2x$$

Step 3: Verify that left hand side is a derivative

$$\frac{d}{dx}\left(e^{3x}y\right) = e^{3x}\frac{dy}{dx} + 3e^{3x}y = 2x$$

Step 4 : Integrate

$$e^{3x}y = \int 2x\,dx = x^2 + C$$

Step 5: Solve for y

$$y = x^2e^{-3x} + Ce^{-3x}$$

If initial condition is given: Step 6 solve for constant C

Say $y(0)=1$. $\quad 1 = 0^2e^{-3\cdot 0} + Ce^{-3\cdot 0} = C$

So $y(x) = x^2e^{-3x} + e^{-3x}$ is particular solution

Another example $\quad x\frac{dy}{dx} - y = x^3$

STANDARD FORM: $\quad \frac{dy}{dx} - \frac{1}{x}y = x^2$

The slope field is undefined at $x=0$, so we only study solutions for $x>0$ OR $x<0$.

Let's assume we are only interested in $\boxed{x>0}$

$P(x) = -\frac{1}{x} \Rightarrow$ integrating factor $e^{\int -\frac{1}{x}dx} = e^{-\ln x} = x^{-1}$

Note that we could write $\int -\frac{1}{x}dx = -\ln x$ because we only care about $x>0$.

Multiply by x^{-1} $\quad x^{-1}\frac{dy}{dx} - x^{-2}y = x^{-1}x^{2} = x$

$$\frac{d}{dx}(x^{-1}y) = x$$

$$x^{-1}y = \frac{1}{2}x^2 + C$$

$$\boxed{y = \frac{1}{2}x^3 + Cx \text{ is the general solution for } x>0.}$$

We can use this method to get formulas that are useful in numerical calculation, even if they are not totally explicit:

Newton's Law of cooling with <u>variable ambient temperature</u>.

Consider $\frac{dT}{dt} = -k(T-A)$ where $A = A(t)$ is a function of time.

One may think we are controlling the temperature in the room where an object with temperature $T(t)$ sits.

Let's also impose an initial condition $T(0) = T_0$

Let's see what the method says:

Standard form: $\frac{dT}{dt} + kT = kA(t)$

Integrating factor $e^{\int k\,dt} = e^{kt}$

Multiply $e^{kt}\frac{dT}{dt} + ke^{kt}T = ke^{kt}A(t)$

$$\frac{d}{dt}\left(e^{kt}T\right) = ke^{kt}A(t)$$

$$e^{kt}T = \int ke^{kt}A(t)\,dt + C$$

$$T = e^{-kt}\int ke^{kt}A(t)\,dt + Ce^{-kt}$$

What about initial condition $T(0) = T_0$?
It's hard to say because we are writing things in terms of indefinite integrals.

We can be more precise if we use definite integrals

Start from $\frac{d}{dt}\left(e^{kt}T\right) = ke^{kt}A(t)$

Now integrate $\int_0^{t_1} - \,dt$ $\quad$ 0 = initial time
t_1 = some later time

$$\int_0^{t_1}\frac{d}{dt}\left(e^{kt}T\right)dt = \int_0^{t_1} ke^{kt}A(t)\,dt$$

$$e^{kt_1}T(t_1) - e^{k\cdot 0}T(0) = \int_0^{t_1} ke^{kt}A(t)dt$$

But $e^{k\cdot 0}T(0) = T(0) = T_0$ which is known.

So $$e^{kt_1}T(t_1) = T_0 + \int_0^{t_1} ke^{kt}A(t)dt$$

$$T(t_1) = T_0 e^{-kt_1} + e^{-kt_1}\int_0^{t_1} ke^{kt}A(t)dt$$

So we have a precise formula in terms of the initial condition T_0 and the definite integral of an expression involving $A(t)$.

A general idea is that using definite integrals makes the management of constants of integration more precise.