Separable equations

Today we will finally learn how to solve the equation

$$\frac{dy}{dx} = xy$$

Can't integrate directly, but what if we rearrange first?

Multiply by $\frac{1}{y}$: $\quad \frac{1}{y}\frac{dy}{dx} = x$

Now integrate both sides $\int dx$:

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int x\,dx = \frac{1}{2}x^2 + C$$

As for $\int \frac{1}{y}\frac{dy}{dx}\,dx$, it equals $\int \frac{1}{y}\,dy$

Why? (1) This is a type of u-substitution, with $u=y$

Consider $u = y$, $du = \frac{dy}{dx}dx$ so $\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{u}\,du = \int \frac{1}{y}\,dy$

Equivalently, we can use the chain rule and the fundamental theorem of calculus:

$$\frac{d}{dx}\left(\int \frac{1}{y}\,dy\right) \overset{\text{Chain rule}}{=} \frac{d}{dy}\left(\int \frac{1}{y}\,dy\right)\frac{dy}{dx} \overset{\text{FTC}}{=} \frac{1}{y}\frac{dy}{dx}$$

So $\int \frac{1}{y}\,dy = \int \frac{1}{y}\frac{dy}{dx}\,dx$

Either way, we know $\int \frac{1}{y}\,dy = \ln|y|$.

So we get

$$\ln|y| = \frac{1}{2}x^2 + C.$$

The last step is to solve for y as a function of x.

Exponentiate $\quad |y| = e^{\left(\frac{1}{2}x^2 + C\right)} = e^{\frac{1}{2}x^2} e^{C} = e^{C} e^{\frac{1}{2}x^2}$

Remove absolute value bars! $\quad y = \pm e^{C} e^{\frac{1}{2}x^2}$

Note that $\pm e^{C}$ is just a constant, call it D:

$$y = De^{\frac{1}{2}x^2}$$

Check your solution:

$$\frac{dy}{dx} = D\frac{d}{dx}\left(e^{\frac{1}{2}x^2}\right) = De^{\frac{1}{2}x^2}\frac{d}{dx}\left(\frac{1}{2}x^2\right) = De^{\frac{1}{2}x^2}x = yx$$

The solution is good.

To summarize: $y(x) = De^{\frac{1}{2}x^2}$ is the general solution of $\frac{dy}{dx} = xy$.

A quicker notation uses differentials

$$\frac{dy}{dx} = xy$$

$$\frac{1}{y}\,dy = x\,dx$$

$$\int \frac{1}{y}\,dy = \int x\,dx \quad \longrightarrow \quad \ln|y| = \frac{1}{2}x^2 + C$$

and so on as before.

Another example: Newton's law of cooling

$$\frac{dT}{dt} = -k(T-A)$$

Separate: $\quad \frac{1}{T-A}\,dT = -k\,dt$

Integrate $\quad \int \frac{1}{T-A}\,dT = \int -k\,dt = -kt + C$

To do $\int \frac{1}{T-A}\,dT$, substitute $u = T-A$, $du = dT$

$$\int \frac{1}{u}\,du = \ln|u| = \ln|T-A|$$

We get $\quad \ln|T-A| = -kt + C$

$$|T-A| = e^{-kt}e^{C}$$

$$T-A = \pm e^{C}e^{-kt}$$

$$T-A = De^{-kt}$$

$$T = A + De^{-kt}$$

D = arbitrary constant.

Another example: $\frac{dy}{dx} = y^2$

$$y^{-2}\frac{dy}{dx} = 1$$

$$\int y^{-2}\frac{dy}{dx}\,dx = \int 1\,dx = x + C$$

$$\|$$

$$\int y^{-2}\,dy = -y^{-1}$$

So $-y^{-1} = x + C$

Solve for y: $y^{-1} = -x - C$

$$y = \frac{-1}{x+C}$$

So $y = \frac{-1}{x+C}$ is a general solution of $\frac{dy}{dx} = y^2$

But observe that $y(x) = 0$ (constantly zero) is a solution:

$$\frac{dy}{dx} = 0 = y^2$$

<u>Notation</u>: triple equals $\equiv$ mean identically equal, that is equal for all values of x (or the independent variable) So $y(x) \equiv 0$ means y is constantly equal to zero.

But there is <u>no way</u> to pick C to get $y(x) \equiv 0$
We will come back to this.

The general method:

Assume that the righthand side of the equation is a function of x times a function of y:

$$\frac{dy}{dx} = f(x,y) = g(x)\, h(y)$$

Divide by $h(y)$ and integrate

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx$$

Give names $F(y) = \int \frac{1}{h(y)}\, dy$, $G(x) = \int g(x)\, dx$

Then we obtain an equation like

$$F(y) = G(x) + C$$

This is an implicit equation for the solutions.

The last step is to solve for y as a function of x.
(Essentially, we need to find the inverse function of $F(y)$)

First, This may or may not be practical.

Example where it isn't practical:

$$\frac{dy}{dx} = \frac{1}{x}\,\frac{y}{1+y} \longrightarrow \frac{1+y}{y}\,dy = \frac{1}{x}\,dx$$

$$\int \left(\frac{1}{y}+1\right) dy = \int \frac{1}{x}\,dx \rightarrow \ln|y| + y = \ln|x| + C$$

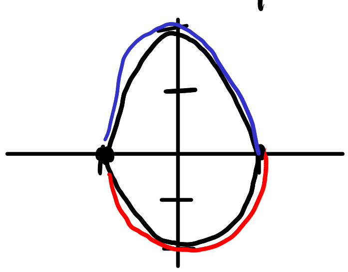
$$|y|\,e^{y} = e^{C}\,|x|$$

Can't solve for y in elementary terms (cf. Lambert W function)

Second, the solution curves of $F(y) = G(x) + C$ may correspond to several <u>branches</u> of the solution of the differential equation.

$$\frac{dy}{dx} = -\frac{4x}{y} \xrightarrow{\text{several steps}} x^2 + \frac{y^2}{4} = C$$

Solutions of $x^2 + \frac{y^2}{4} = C$ are ellipses, but only upper or lower half is a valid solution (since y must be a function of x



Third, some solutions may not be captured by the general form you find, e.g. $y(x) \equiv 0$ for $\frac{dy}{dx} = y^2$

Such a solution is called a singular solution.

Q Consider $\frac{dy}{dx} = y$. What's the difference between

$$y = \pm e^C e^x \quad \text{and} \quad y = D e^x \; ?$$

A: $\pm e^C$ can be any positive or negative number, whereas D can be any positive or negative number $\boxed{\text{or zero}}$

While both $y = \pm e^C e^x$ and $y = D e^x$ are general solutions (because they contain undetermined constants), the latter is more general because it includes the case $D = 0$.

We usually want the most general solution we can find.