Sturm-Liouville theory 2: Eigenfunction Expasions

Sturm-Liouville problem on $[a,b]$

(1) $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x) y = 0$

(2) $\alpha_1 y(a) - \alpha_2 y'(a) = 0$

(3) $\beta_1 y(b) + \beta_2 y'(b) = 0$

If $p(x) > 0$ and $r(x) > 0$ on $[a,b]$,
We know that the eigenvalues λ form an increasing sequence.

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$$

and that the associated eigenfunctions

$$y_1(x), y_2(x), \ldots, y_n(x), \ldots$$

are unique up to multiplication by a constant.

They also satisfy the orthogonality relation

$$\boxed{\text{If } n \neq m, \text{ then } \int_a^b y_n(x) y_m(x) r(x)\, dx = 0}$$

The last part of the story is that (pretty much) any function $f(x)$ defined on $[a,b]$ can be written as a series in the Eigenfunctions:

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

This is called the eigenfunction series for $f(x)$ coming from the Sturm-Liouville problem.

Thus for each function $f(x)$, there are many ways to write it as a series, one for each Sturm-Liouville problem.

Key examples: ① (1) $y'' + \lambda y = 0$ $\quad (p=1, q=0, r=1)$

(2) $y(0) = 0$ $\quad (\alpha_1 = 1, \alpha_2 = 0)$

(3) $y(L) = 0$ $\quad (\beta_1 = 1, \beta_2 = 0)$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad y_n(x) = \sin\frac{n\pi x}{L} \qquad n = 1, 2, 3, \ldots$

Orthogonality: $\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = 0$ if $n \neq m$

Eigenfunction Series: $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

where $b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$ [Sine series]

Ⅱ

(1) $y'' + \lambda y = 0$ $\quad (p=1, q=0, r=1)$

(2) $y'(0) = 0$ $\quad (\alpha_1 = 0, \alpha_2 = 1)$

(3) $y'(L) = 0$ $\quad (\beta_1 = 0, \beta_2 = 1)$

$\lambda_0 = 0, \ \lambda_1 = \left(\frac{\pi}{L}\right)^2, \ldots, \lambda_n = \left(\frac{n\pi}{L}\right)^2$ $\qquad n = 0, 1, 2, \ldots$

$y_0(x) = 1 \quad y_1(x) = \cos\frac{\pi x}{L}, \ldots, y_n = \cos\frac{n\pi x}{L}$

Orthogonality: $\int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = 0$ if $n \neq m$

Eigenfunction series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$

where $a_0 = \frac{2}{L}\int_0^L f(x)dx, \ a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx$ [cosine series]

In general, how do we find the coefficients of the eigenfunction series

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) \quad ?$$

Using orthogonality relations to extract a single coefficient.

Consider $\displaystyle\int_a^b f(x)\, y_m(x)\, r(x)\, dx = \int_a^b \left(\sum_{n=1}^{\infty} c_n y_n(x) \right) y_m(x)\, r(x)\, dx$

$$= \int_a^b \sum_{n=1}^{\infty} c_n y_n(x) y_m(x) r(x)\, dx = \sum_{n=1}^{\infty} c_n \int_a^b y_n(x)\, y_m(x)\, r(x)\, dx$$

by orthogonality only the $n=m$ term can be nonzero, so

$$= c_m \int_a^b y_m(x) y_m(x)\, r(x)\, dx.$$

In summary: $\displaystyle\int_a^b f(x)\, y_m(x)\, r(x)\, dx = c_m \int_a^b [y_m(x)]^2 r(x)\, dx$

Or

$$\boxed{c_m = \frac{\int_a^b f(x) y_m(x)\, r(x)\, dx}{\int_a^b [y_m(x)]^2 r(x)\, dx}}$$

This generalizes the formulas for b_n and a_n in the sine and cosine series.

Another example: (1) $y'' + \lambda y = 0$ $\qquad p = 1 > 0,\ q = 0 \geq 0,\ r = 1 > 0$

(2) $y(0) = 0$ $\qquad \alpha_1 = 1 \geq 0,\ \alpha_2 = 0 \geq 0$

(3) $y'(L) = 0$ $\qquad \beta_1 = 0 \geq 0,\ \beta_2 = 1 \geq 0$

By main theorem of Sturm Liouville theory, all eigenvalues are nonnegative.

$\underline{\lambda=0}$: $y = Ax+B$ $\quad y(0)=0 \Rightarrow B=0 \quad y'(L)=0 \Rightarrow A=0$

$\lambda=0$ is not an eigenvalue

$\underline{\lambda=\alpha^2>0}$: $\quad y = A\cos\alpha x + B\sin\alpha x \qquad y(0)=0 \Rightarrow A=0$

so $y = B\sin\alpha x$

$y'(x) = B\alpha\cos\alpha x \qquad y'(L)=0 \Rightarrow B\alpha\cos\alpha L = 0$

so to be an eigenvalue, must have $\cos\alpha L = 0$.

Thus $\alpha L = \left(n-\frac{1}{2}\right)\pi = \left(\frac{2n-1}{2}\right)\pi \qquad n=1,2,3,\dots$

$$\alpha_n = \left(\frac{2n-1}{2}\right)\frac{\pi}{L} \qquad \lambda_n = \alpha_n^2 = \left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}\right]^2$$

Eigenfunctions $\quad y_n(x) = \sin\alpha_n x = \sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right]$

Orthogonality: If $n \neq m$, $\displaystyle\int_0^L \sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right]\sin\left[\left(\frac{2m-1}{2}\right)\frac{\pi}{L}x\right]dx = 0$

Eigenfunction series

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right]$$

Where

$$c_n = \frac{\int_0^L f(x)\sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right]dx}{\int_0^L \sin^2\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right]dx}$$

This is the so-called odd-half multiple sine series for $f(x)$.
It comes up, for instance in the heat problem.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad u(0,t)=0 \qquad \frac{\partial u}{\partial t}(0,t) = 0$$

$$u(x,0) = f(x)$$

where the $x=0$ end is held fixed at zero temp, but the $x=L$ end is insulated.