Sturm–Liouville problems and eigenfunction series

In heat, wave, and Laplace's equation, separation of variables often leads to problems like

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

The eigenfunctions are $\sin \frac{n\pi x}{L}$.

Usually the initial data has to be expressed as a sine series. Can we generalize this?

Sturm–Liouville equations:

A second order linear equation with parameter λ is in Sturm–Liouville standard form if it looks like

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x) y = 0 \qquad (\text{SL form})$$

Fact: any second order linear equation

$$A(x)y'' + B(x)y' + C(x)y + \lambda D(x) y = 0$$

Can be put in SL standard form

Main example: $\frac{dy^2}{dx^2} + \lambda y = 0$ $\qquad p(x) = 1 \quad q(x) = 0$, $r(x) = 1$

is in SL standard form.

Other example: $\frac{d}{dx}\left[x\frac{dy}{dx}\right] - \frac{n^2}{x}y + \lambda x y = 0$

Bessel equation (Laplace eqn in cylindrical coords)

Endpoint conditions: Domain $[a,b]$

At $x=a$: $\alpha_1 y(a) - \alpha_2 y'(a) = 0$

At $x=b$: $\beta_1 y(b) + \beta_2 y(b) = 0$

$\left[\text{Assume } \begin{array}{l} \alpha_1^2+\alpha_2^2 > 0 \\ \beta_1^2+\beta_2^2 > 0 \end{array}\right]$

Sturm-Liouville problem: To determine those pairs $(\lambda, y(x))$ satisfying

(1) $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x) y = 0$

(2) $\alpha_1 y(a) - \alpha_2 y'(a) = 0$

(3) $\beta_1 y(b) - \beta_2 y(b) = 0$

Obviously $y(x) = 0$ and λ = anything is a solution.
The values of λ for which a solution $y(x)$ not constant $=0$ exists are called eigenvalues. The corresponding functions $y(x)$ are called eigenfunctions.

We consider this problem so that we can state a general theorem.

Theorem 1 (a) Suppose that $p(x), p'(x), q(x), r(x)$ are continuous on $[a,b]$
And $p(x) > 0$ and $r(x) > 0$ for all points in $[a,b]$.
Then the eigenvalues form an increasing sequence

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty.$$

The eigenfunction $y_n(x)$ associated to λ_n is unique up to multiplication by a constant.

(b) Suppose additionally that $q(x) \geq 0$ on $[a,b]$, and that the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ are nonnegative.
Then all eigenvalues are nonnegative $\lambda_n \geq 0$.

Illustration: $y'' + \lambda y = 0 \quad (0 < x < L)$

$$y(0) = 0 \ , \quad h y(L) + y'(L) = 0 \quad (h > 0).$$

This satisfies the hypotheses of the theorem: $p(x) = 1 > 0$, $r(x) = 1 > 0$ so it satisfies clause (a). Also $q(x) = 0 \geq 0$ and $\alpha_1 = 1 \geq 0$, $\alpha_2 = 0 \geq 0$ $\beta_1 = h \geq 0$, $\beta_2 = 1 \geq 0$ so it satisfies clause (b).
Thus we know from the theorem that there are no negative eigenvalues.

$\underline{\lambda = 0}$: $\quad y = Ax + B \ , \quad y(0) = 0 \Rightarrow B = 0$.

$$h y(L) + y'(L) = hAL + A = (hL+1)A \Rightarrow A = 0$$

So $\lambda = 0$ is not an eigenvalue

$\underline{\lambda > 0}$: Write $\lambda = \alpha^2$, so $y(x) = A\cos\alpha x + B\sin\alpha x$

$$y(0) = 0 \Rightarrow A = 0 \quad \text{so} \quad y(x) = B\sin\alpha x$$

Now
$$\begin{aligned} 0 = h y(L) + y'(L) &= hB\sin\alpha L + \alpha B\cos\alpha L \\ &= B(h\sin\alpha L + \alpha\cos\alpha L) \end{aligned}$$

B will be forced to be zero unless

$$h\sin\alpha L + \alpha\cos\alpha L = 0$$

$$h\sin\alpha L = -\alpha\cos\alpha L$$

$$h\tan\alpha L = -\alpha$$

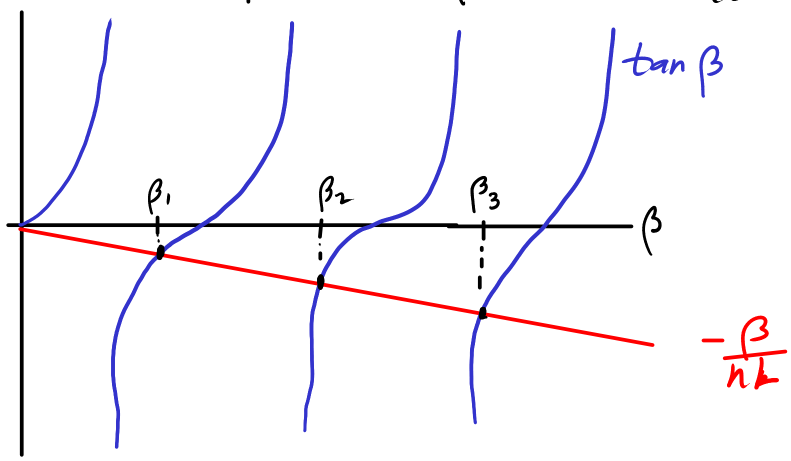
$$\tan\alpha L = -\frac{\alpha}{h}$$

Set $\beta = \alpha L$, the equation becomes

$$\tan\beta = -\frac{\beta}{hL}$$

This is a transcendental equation that cannot be solved analytically (as far as I know)

To see solutions, plot $\tan\beta$ and $\frac{\beta}{hL}$



Let β_n denote the nth positive root of the equation
$$\tan\beta = -\frac{\beta}{hL}$$

Thus $\lambda_n = \alpha_n^2 = (\beta_n/L)^2$ are the eigenvalues and
$y_n(x) = \sin\alpha_n x = \sin\frac{\beta_n x}{L}$ are the eigenfunctions.

<u>Theorem 2</u> Suppose that a Sturm-Liouville problem satisfies the hypotheses of Theorem 1(a). Then the eigenfunctions $y_n(x)$ satisfy
$$\int_a^b y_n(x)\, y_m(x)\, r(x)\, dx = 0 \quad \text{if} \quad n \neq m.$$

[In words, eigenfunctions for distinct eigenvalues are orthogonal with respect to the weighted inner product $\langle f, g\rangle = \int_a^b f(x)g(x)r(x)\,dx$]

<u>Illustration</u>: Let $y_n = \sin\frac{\beta_n x}{L}$, $0<x<L$ be the eigenfunctions

Then if $n \neq m$, $\int_0^L \sin\left(\frac{\beta_n x}{L}\right) \sin\frac{\beta_m x}{L}\, dx = 0$

This is interesting, because we know this even though we don't know what β_n and β_m are exactly.