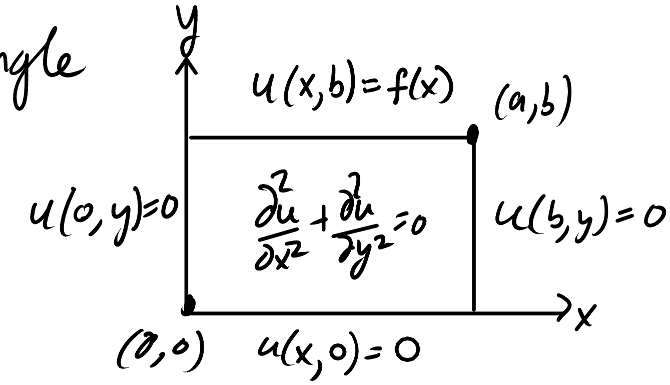
Laplace's Equation on a rectangle

Today we will solve the problem



Homogeneous boundary conditions on all sides except one.

Separation of variables: Separable solutions $u(x,y) = X(x)\,Y(y)$

Laplace eqn: $$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{d^2 X}{dx^2}\,Y + X\,\frac{d^2 Y}{dy^2}$$

Divide by XY: $$0 = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}$$

So $$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{only depends on } x} = \underbrace{-\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{only depends on } y}$$

& $\Longrightarrow$ both sides constant $= -\lambda$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda = -\frac{1}{Y}\frac{d^2Y}{dy^2} \Longrightarrow \begin{aligned} \frac{d^2X}{dx^2} + \lambda X &= 0 \\ \frac{d^2Y}{dy^2} - \lambda Y &= 0 \end{aligned}$$

Boundary conditions:
$$u(x,0) = 0$$
$$u(x,b) = f(x)$$
$$u(0,y) = 0$$
$$u(a,y) = 0$$

Let's forget about this one for awhile. We first work with the homogeneous conditions.

$$u(0,y)=0 \Rightarrow X(0)\,Y(y)=0 \Rightarrow X(0)=0$$
$$u(a,y)=0 \Rightarrow X(a)\,Y(y)=0 \Rightarrow X(a)=0$$
$$u(x,0)=0 \Rightarrow X(x)\,Y(0)=0 \Rightarrow Y(0)=0$$

So we get
$$\underbrace{\frac{d^2X}{dx^2}+\lambda X=0 \qquad X(0)=0 \qquad X(a)=0}_{}$$
$$\frac{d^2Y}{dy^2}-\lambda Y=0 \qquad Y(0)=0$$

That same eigenvalue problem!

Eigenvalues: $\lambda_n=\left(\frac{n\pi}{a}\right)^2 \quad n=1,2,3,\ldots$

Eigenfunctions: $X_n(x)=\sin\frac{n\pi x}{a}$

What is the Y_n- partner for X_n?

$\lambda_n=\left(\frac{n\pi}{a}\right)^2$ so Y_n satisfies $\frac{d^2Y_n}{dy^2}-\left(\frac{n\pi}{a}\right)^2Y_n=0$

characteristic eqn $r^2-\left(\frac{n\pi}{a}\right)^2=0 \Rightarrow r=\pm\frac{n\pi}{a}$

so $Y_n(y)=c_1e^{\frac{n\pi y}{a}}+c_2e^{-\frac{n\pi y}{a}}$

We can now use the condition $Y(0)=0$:
$$0=Y_n(0)=c_1e^0+c_2e^0=c_1+c_2$$

Thus $c_2=-c_1$, and we can write
$$Y_n=c_1\left(e^{\frac{n\pi y}{a}}-e^{-\frac{n\pi y}{a}}\right)=2c_1\left(\frac{e^{\frac{n\pi y}{a}}-e^{-\frac{n\pi y}{a}}}{2}\right)$$
$$=2c_1\sinh\left(\frac{n\pi y}{a}\right) \quad \text{hyperbolic sine.}$$

! So $u_n(x,y) = \sinh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$ satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$,

and $u_n(0,y)=0$, $u_n(a,y)=0$, $u_n(x,0)=0$!

General solution of Laplace eqn and the three homogeneous boundary conditions:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$$

2 NOTA BENE: It makes a difference which side has the nonhomogeneous boundary condition !!!

Continuing: We still need to satisfy $u(x,b) = f(x)$

$$u(x,b) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right) \overset{?}{=} f(x)$$

Again this is a sine series for $f(x)$, with $L=a$:

$$c_n \sinh\left(\frac{n\pi b}{a}\right) = b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

So $$c_n = \left[\sinh\left(\frac{n\pi b}{a}\right)\right]^{-1}\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

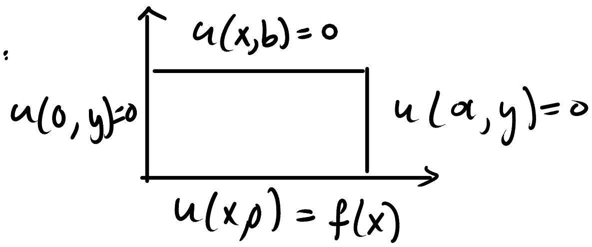
<u>Example</u>: if $f(x) = x$, then

$$b_n = \frac{2}{a}\int_0^a x\sin\frac{n\pi x}{a}\,dx = \frac{2a(-1)^{n+1}}{n\pi} \Rightarrow f(x) = x = \sum_{n=1}^{\infty}\frac{2a(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{a}$$

Thus $c_n = \left[\sinh\left(\frac{n\pi b}{a}\right)\right]^{-1} \frac{2a(-1)^{n+1}}{n\pi}$

So $u(x,y) = \sum_{n=1}^{\infty} \left[\sinh\left(\frac{n\pi b}{a}\right)\right]^{-1} \frac{2a(-1)^{n+1}}{n\pi} \sinh\left(\frac{n\pi y}{a}\right) \sin\frac{n\pi x}{a}$

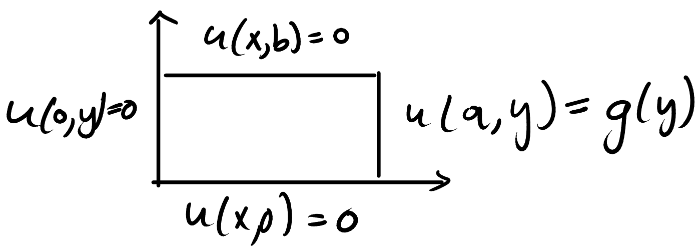
OTHER CASES:



$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi(b-y)}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

$$u(x,0) = \sum c_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right) = f(x)$$

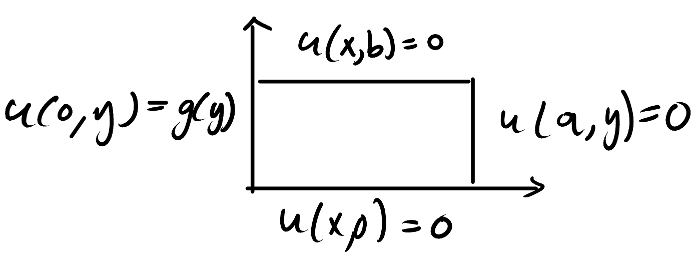
$$c_n = \left[\sinh\frac{n\pi b}{a}\right]^{-1} \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$$



$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$u(a,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right) = g(y)$$

$$c_n = \left[\sinh\frac{n\pi a}{b}\right]^{-1} \frac{2}{b}\int_0^b g(y) \sin\frac{n\pi y}{b}\, dy$$



$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi(a-x)}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$u(0,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right) = g(y)$$

$$c_n = \left[\sinh\left(\frac{n\pi a}{b}\right)\right]^{-1} \frac{2}{b}\int_0^b g(y) \sin\frac{n\pi y}{b}\, dy$$