Laplace equation I

We consider functions of two variables $u(x,y)$, where now both x and y are thought of as <u>spatial</u> variables.

The <u>two-dimensional Laplace equation</u> is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solutions of Laplace equation are also called <u>harmonic functions</u>.

Similarly, the <u>three-dimensional Laplace equation</u> is for a function $u(x,y,z)$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Note that

$$\operatorname{div}(\operatorname{grad} u) = \nabla \cdot (\nabla u) = \nabla \cdot \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Also written $\nabla^2 u$ or Δu, called the <u>Laplacian</u> of u.
The operator

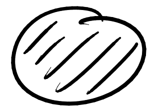
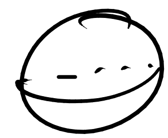
$$\Delta = \nabla^2 = \operatorname{div}\operatorname{grad} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called the <u>Laplacian</u> or <u>Laplace operator</u>.

In 2d: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In 1d: $\Delta = \frac{d^2}{dx^2}$

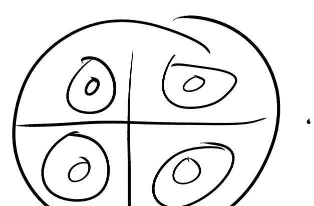
This is one of the most important operators in mathematics.

The heat equation in 2 or 3 spatial dimensions and 1 time dimension is $\frac{\partial u}{\partial t} = k \Delta u$

Think of heat in plate  or ball 

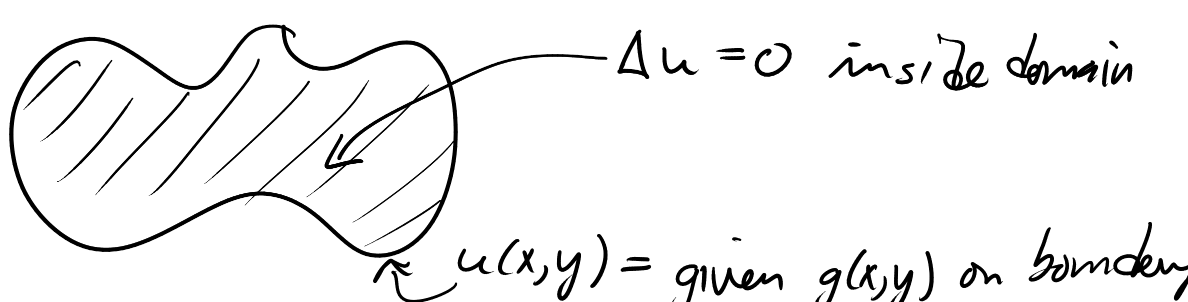
The wave equation in 2 or 3 spatial dimensions and 1 time dim.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

Think vibrations in a drum head 

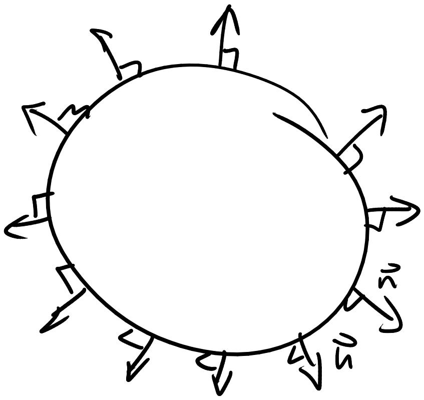
Steady state temperature: $\frac{\partial u}{\partial t} = k \Delta u$ and $\frac{\partial u}{\partial t} = 0$,

Hence $\Delta u = 0$. So Laplace's equation describes Temperature distribution in equilibrium. But we still need a boundary condition: We can specify u on the boundary of the domain



These are called <u>Dirichlet</u> boundary conditions (keep temperature on the boundary fixed.)

There are also Neumann boundary conditions, where we require $\vec{n} \cdot \nabla u = 0$ along boundary, where $\vec{n}$ is the normal vector to the boundary:



Require directional derivative $\vec{n} \cdot \nabla u = 0$
In heat case this corresponds to insulated boundary.

Other applications: Electrostatic potential $V(x,y,z)$

Electric field $\vec{E} = \nabla V$.

Gauss' law $\nabla \cdot \vec{E} = \rho$ where ρ is charge density.

If $\rho = 0$:

$$0 = \nabla \cdot \vec{E} = \nabla \cdot (\nabla V) = \nabla^2 V = \Delta V$$

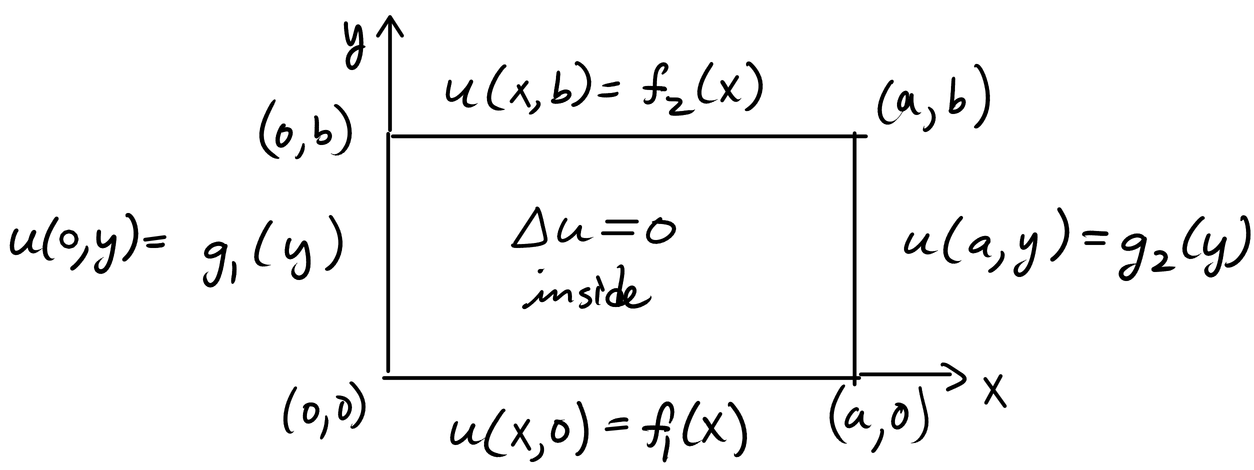
So V satisfies Laplace's equation in regions where no charge is present.

Example situation:

V is specified on boundary of sphere

$\Delta V = 0$ inside.

In this course we will solve Laplace's equation on a rectangle with Dirichlet boundary conditions.



These boundery conditions are very much nonhomogeneous, So we can't apply separation of variables directly.

On the other hand for homogeneous boundary conditions the only solution is $u=0$.

$u=0$ (top)

$u=0$ | $\Delta u = 0$ | $u=0$

$u=0$ (bottom)

We want to reduce to the case where all but one of the four sides carries a homogeneous boundary condition.

If u_L solves

$u_L=0$ (top)

$u_L=g_1(y)$ | $\Delta u_L=0$ | $u_L=0$ (left)

$u_L=0$ (bottom)

If u_R solves

$u_R=0$ (top)

$u_R=0$ | $\Delta u_R=0$ | $u_R=g_2(y)$ (right)

$u_R=0$ (bottom)

If u_T solves

$u_T=f_2(x)$ (top)

$u_T=0$ | $\Delta u_T=0$ | $u_T=0$ (top)

$u_T=0$ (bottom)

If u_B solves

$u_B=0$ (top)

$u_B=0$ | $\Delta u_B=0$ | $u_B=0$ (bottom)

$u_B=f_1(x)$ (bottom)

Then $u = u_L + u_R + u_T + u_B$ solves

$u=f_2(x)$ (top)

$u=g_1(y)$ | $\Delta u=0$ | $u=g_2(y)$!

$u=f_1(x)$ (bottom)