Wave equation: For waves in water, air, vibrating string, ...

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(x,t) =$ water level, air pressure, displacement of vibrating medium.

We will solve the vibrating string using separation of variables.

But first:

To break up the monotony a bit, let's look at the wave eqn without boundary conditions, i.e., x ranges over the whole real line.

There is an amazingly elegant solution found by D'Alembert:

Let $F(z)$ and $G(z)$ be functions (assume F'' and G'' exist)

Then

$$u(x,t) = F(x-ct) + G(x+ct)$$

Satisfies $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Check it: $$\frac{\partial u}{\partial t} = -cF'(x-ct) + cG'(x+ct)$$

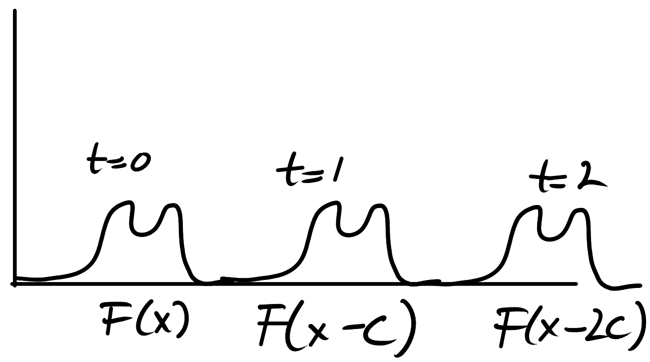
$$\frac{\partial^2 u}{\partial t^2} = c^2 F''(x-ct) + c^2 G''(x+ct)$$

$$\frac{\partial u}{\partial x} = F'(x-ct) + G'(x+ct)$$

$$\frac{\partial^2 u}{\partial x^2} = F''(x-ct) + G''(x+ct)$$

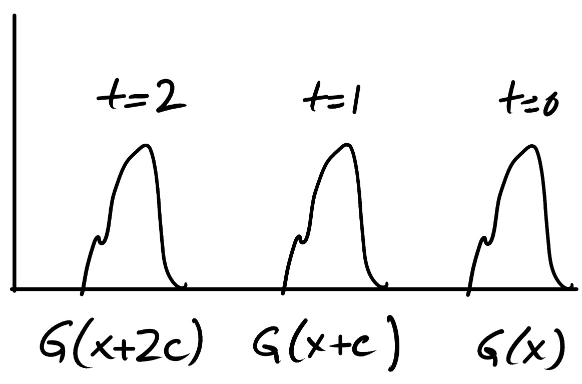
this equals c^2 times this

What is $F(x-ct)$? Shape stays the same, but it translates/shifts to the right at speed c!



$F(x-ct)$ is called a <u>right-moving wave</u>.

What is $G(x+ct)$? Shape stays the same, but it translates/shifts to the left at speed c!



$G(x+ct)$ is called a <u>left-moving wave</u>.

The general solution is a superposition of right- and left-moving components.

The parameter c is the speed of waves.

For Electromagnetic waves in vacuum, $c = 299\,792\,458$ m/s

The wave equation is $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

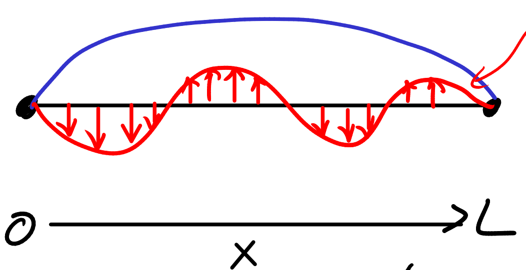
c is the speed of the wave; it depends on the medium.

Free wave propagation

$$u(x,t) = \underbrace{F(x-ct)}_{\text{right moving wave}} + \underbrace{G(x+ct)}_{\text{left moving wave}}$$

These waves have no boundary conditions, and the pass right through each other without interacting. They are "free".

We will be interested in a situation where the waves are confined, such as in a vibrating string:



$u(x,t)$ represents displacement of string from its rest position.

(L = length)

The ends of the string are fixed. They don't get displaced. This translates into the boundary conditions

$$u(0,t) = 0 \quad , \quad u(L,t) = 0$$

The function $u(x,t)$ will satisfy the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Since $u(x,t)$ represents the position, $\frac{\partial u}{\partial t}$ is the velocity and $\frac{\partial^2 u}{\partial t^2}$ is the acceleration. So the wave equation says that the acceleration depends on the shape of the string. (Recall that $\frac{\partial^2 u}{\partial x^2}$ measures the convexity / concavity of the graph.)

Since the differential equation is telling us acceleration, we need to specify the initial position and initial velocity in order to get a unique solution.

Initial conditions: $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$

where $f(x)$ describes initial position
and $g(x)$ describes initial velocity.

All together, we get a complete wave equation problem

$$\left\{\begin{array}{ll} \text{Domain } 0 \le x \le L & \\ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, & u(x,0) = f(x), \\ & \frac{\partial u}{\partial t}(x,0) = g(x). \\ u(0,t) = 0, & \\ u(L,t) = 0, & \end{array}\right\}$$

For any reasonable functions $f(x)$ and $g(x)$, this problem has a unique solution $u(x,t)$.