

Heat Equation III

Today we will finish solving the problem

$$\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Recap: so far, we found functions $u(x,t) = \bar{X}(x)T(t)$ that satisfy the heat equation and the boundary cond. This translates into

$$\begin{cases} \frac{dT}{dt} = -k\lambda T & \frac{d^2\bar{X}}{dx^2} + \lambda\bar{X} = 0 \\ & \bar{X}(0) = 0 \\ & \bar{X}(L) = 0 \end{cases} \left. \vphantom{\begin{cases} \frac{dT}{dt} = -k\lambda T \\ \frac{d^2\bar{X}}{dx^2} + \lambda\bar{X} = 0 \\ \bar{X}(0) = 0 \\ \bar{X}(L) = 0 \end{cases}} \right\} \text{eigenvalue problem}$$

Eigenvalues and Eigenfunctions:

$$\begin{cases} \lambda_n = \left(\frac{n\pi}{L}\right)^2 \\ \bar{X}_n(x) = \sin \frac{n\pi x}{L} \end{cases} \left. \vphantom{\begin{cases} \lambda_n = \left(\frac{n\pi}{L}\right)^2 \\ \bar{X}_n(x) = \sin \frac{n\pi x}{L} \end{cases}} \right\} n=1, 2, 3, \dots$$

Corresponding $T(t)$'s: $T_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

Basic solutions $u_n(x,t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi x}{L}$

$$u_n(x,t) \text{ satisfies: } \begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases} \left. \vphantom{\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases}} \right\} \begin{array}{l} \text{But it doesn't} \\ \text{satisfy initial} \\ \text{condition!} \\ u(x,0) = f(x) \end{array}$$

In general, none of the functions $u_n(x,t)$ will satisfy initial condition. We need a more general solution!

Idea: consider

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

Proposition: This $u(x,t)$ satisfies $\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ u(L,t) = 0 \end{array} \right\}$

Proof: $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \left(\sum_{n=1}^{\infty} c_n u_n \right) = \sum_{n=1}^{\infty} c_n \frac{\partial u_n}{\partial t} = \sum_{n=1}^{\infty} c_n k \frac{\partial^2 u_n}{\partial x^2}$ b/c u_n is a solution

$$= k \frac{\partial^2}{\partial x^2} \left(\sum_{n=1}^{\infty} c_n u_n \right) = k \frac{\partial^2 u}{\partial x^2} \quad \checkmark$$

$$u(0,t) = \sum_{n=1}^{\infty} c_n u_n(0,t) = \sum_{n=1}^{\infty} c_n \cdot 0 = 0 \quad \checkmark$$

$$u(L,t) = \sum_{n=1}^{\infty} c_n u_n(L,t) = \sum_{n=1}^{\infty} c_n \cdot 0 = 0 \quad \checkmark$$

This proposition is a superposition principle.

So now we have a general solution which is a series

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n = \sum_{n=1}^{\infty} c_n e^{-k \left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi x}{L}$$

As with any general solution, we can determine the constants c_n from the initial data:

We want to match $u(x,0)$ with some given function $f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}$$

Observe: This is a Fourier sine series of period $2L$!

We want it to equal $f(x)$ for $0 < x < L$

WANT : $f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}$, $0 < x < L$
 TO MATCH:

This is precisely the meaning of "the Fourier sine series of $f(x)$ ". So we know how to find the coefficients c_n :

Conceptually: extend $f(x)$ to an odd periodic function of period $2L$; take Fourier series

Formulaically: $c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

If we want to, we can write a big honking formula for the solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \right] e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi x}{L}$$

solves (1) $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$, (2) $u(0,t) = 0$, (3) $u(L,t) = 0$, and

(4) $u(x,0) = f(x)$

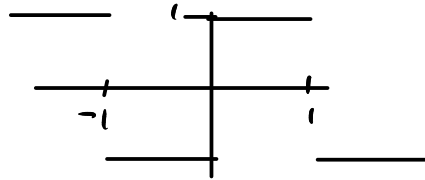
Example: $k=3$; $L=1$; $f(x)=1$, $0 < x < 1$

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = 3 \frac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = 1, 0 < x < 1 \end{array} \right\} \begin{array}{l} \lambda_n = (n\pi)^2 \quad X_n = \sin n\pi x \\ u_n = e^{-3(n\pi)^2 t} \sin n\pi x \\ u = \sum_{n=1}^{\infty} C_n e^{-3(n\pi)^2 t} \sin n\pi x \end{array}$$

Want to write $f(x) = 1 = \sum_{n=1}^{\infty} C_n \sin n\pi x$

Odd periodic extension

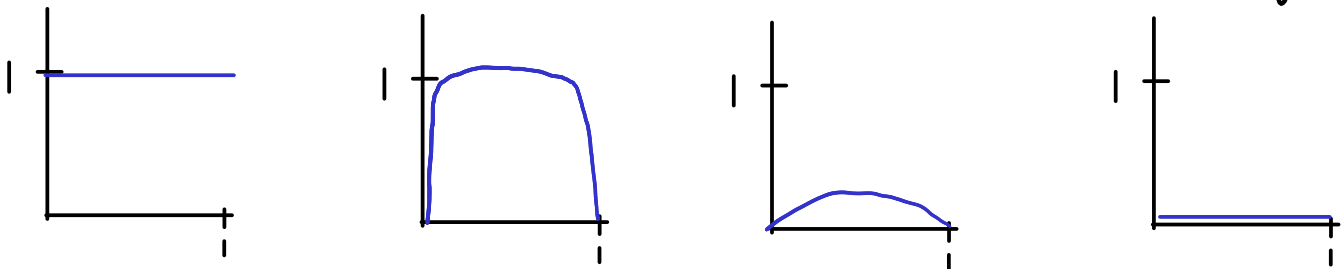
Square wave!



$$f(x) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin n\pi x \implies C_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

So $u(x,t) = \sum_{n \text{ odd}} \frac{4}{n\pi} e^{-3(n\pi)^2 t} \sin n\pi x$ solves all parts of the problem.

Initial state, Instant later, while later, eventually



The higher frequencies decay faster

$$\lim_{t \rightarrow \infty} e^{-k \left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi x}{L} = 0$$