Heat Equation II

Still trying to solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$. Last time, we sought factorizable solutions of the form $u(x,t) = X(x)T(t)$

$$\frac{dT}{dt}(t)\,X(x) = k\,\frac{d^2X}{dx^2}(x)\,T(t)$$

$$\frac{1}{kT(t)}\frac{dT}{dt}(t) = \frac{1}{X(x)}\frac{d^2X}{dx^2}(x)$$

Recall: LHS doesn't depend on x, RHS doesn't depend on t, so both are constant. We call the constant $-\lambda$.

$$\frac{1}{kT}\frac{dT}{dt} = -\lambda = \frac{1}{X}\frac{d^2X}{dx^2}$$

i.e. $\frac{d^2X}{dx^2} = -\lambda X$ and $\frac{dT}{dt} = -k\lambda T$

i.e. $\left\{\begin{array}{l} \frac{d^2X}{dx^2} + \lambda X = 0 \\ \frac{dT}{dt} = -k\lambda T \end{array}\right\}$ for some value of λ.

These are ordinary differential equations and we know how to solve them!

$$\frac{dT}{dt} = -k\lambda T \implies T(t) = Ce^{-k\lambda t} \quad (C \text{ constant})$$

The solution of $\frac{d^2 X}{dx^2} + \lambda X = 0$ depend on whether $\lambda<0, \lambda=0, \lambda>0$

$\lambda > 0$: $X(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

$\lambda = 0$: $X(x) = A + Bx$

$\lambda < 0$: $X(x) = A e^{\sqrt{-\lambda}x} + B e^{-\sqrt{-\lambda}x}$

Now, $X(x)\,T(t)$ is a solution of the heat equation!

so: if $\lambda > 0$: $u = e^{-k\lambda t}(A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x)$

satisfies $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

if $\lambda = 0$: $u = e^{-k0t}(A + Bx) = A + Bx$

satisfies it: infact, for these solutions

$\frac{\partial u}{\partial t} = 0$ "steady-state solution"

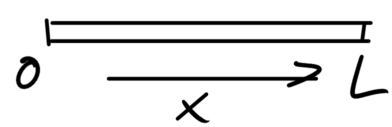
if $\lambda < 0$, let $a^2 = -\lambda$: Then

$u = e^{ka^2t}(Ae^{ax} + Be^{-ax})$ solves it.

these solutions go to $\pm\infty$ as $t \to \infty$, so they don't have much physical meaning.

This is really great: we have lots of solutions to play with.
There is even a free parameter λ that we can vary.
We have too many! Use boundary conditions to cut them down

Let's consider the Rod of length L



* End points held fixed at temperature 0:

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} \text{ Boundary conditions}$$

In terms of the factorizable function $u(x,t) = X(x)T(t)$

This means $u(0,t) = X(0)T(t) = 0$

$u(L,t) = X(L)T(t) = 0$

So we need $\left.\begin{array}{l} X(0)=0 \\ X(L)=0 \end{array}\right\}$ Endpoint conditions for $X(x)$

Thus X satisfies the endpoint value problem

$$\left\{\begin{array}{ll} \dfrac{d^2X}{dx^2} + \lambda X = 0 & (1) \\ X(0) = 0 & (2) \\ X(L) = 0 & (3) \end{array}\right\}$$ This is the eigenvalue problem we studied earlier!

Recall how to find positive eigenvalues and eigenfunctions

$$\frac{d^2X}{dx^2} + \lambda X = 0 \implies X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

$$X(0)=0 \implies A=0 \implies X(x) = B\sin\sqrt{\lambda}x$$

$$X(L)=0 \implies B\sin\sqrt{\lambda}L = 0$$

X can only be "interesting"/nontrivial if $B \neq 0$, so we must have $\sin\sqrt{\lambda}L = 0 \implies \sqrt{\lambda}L = n\pi \quad n=1,2,3,\ldots$

$$\implies \lambda = \left(\frac{n\pi}{L}\right)^2 \quad n=1,2,3,\ldots$$

It turns out $\lambda < 0$ and $\lambda = 0$ are <u>not</u> eigenvalues.

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n=1,2,3,\ldots$

Eigenfunctions: $X_n(x) = \sin\frac{n\pi x}{L}$

Each eigenfunction yields a solution to the heat equation

$$X_n(x) = \sin \frac{n\pi x}{L} \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

What is the T counterpart for X_n?

It should satisfy

$$\frac{dT}{dt} = -k\lambda_n T = -k\left(\frac{n\pi}{L}\right)^2 T$$

Therefore

$$T(t) = Ce^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad C \text{ constant}$$

Put it together:

$$u(x,t) = T(t)X_n(x) = Ce^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi x}{L}$$

Satisfies
$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} & \text{heat eqn} \\ u(0,t) = 0 & \text{Boundary} \\ u(L,t) = 0 & \text{conditions} \end{cases}$$

Another type of boundary condition "insulated ends" $\dfrac{\partial u}{\partial x}(0,t) = 0 \qquad \dfrac{\partial u}{\partial x}(L,t) = 0$

$$\left(\frac{\partial u}{\partial x} = \text{heat flux} \right) \qquad \frac{dX}{dx}(0)\,T(t) = 0 \qquad \frac{dX}{dx}(L)\,T(t) = 0$$

So $X(x)$ satisfies
$$\begin{Bmatrix} X'' + \lambda X = 0 \\ X'(0) = 0 \\ X'(L) = 0 \end{Bmatrix}$$

Eigenvalues: $\lambda_0 = 0$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1,2,3,\ldots$

Eigenfunctions: $X_0 = 1$, $X_n = \cos \frac{n\pi x}{L} \quad n = 1,2,3,\ldots$

Heat solutions: $u_0(x,t) = 1$, $u_n(x,t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos \frac{n\pi x}{L}$