Heat Equation I

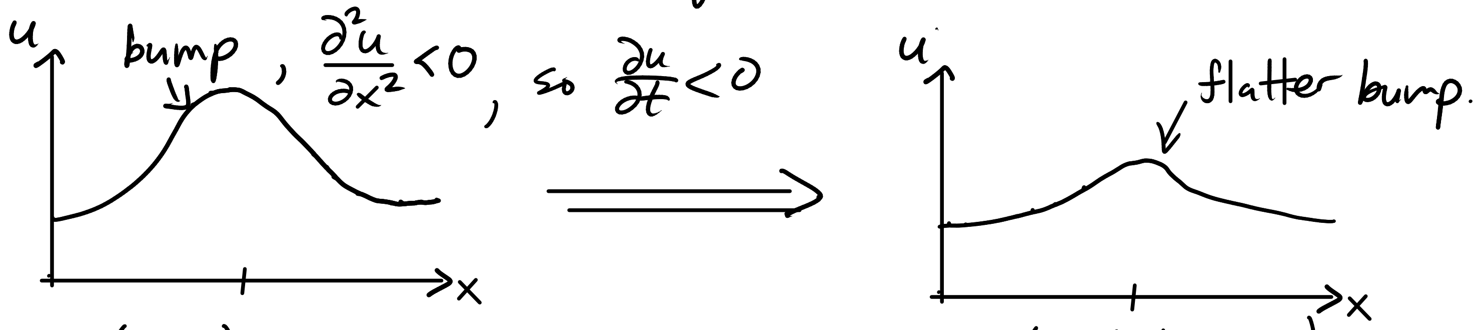
We consider a function $u(x,t)$. The independent variables are x (spatial position) and t (time). The dependent variable u represents temperature.

- $u(x,t)$ satisfies the <u>heat equation</u> if

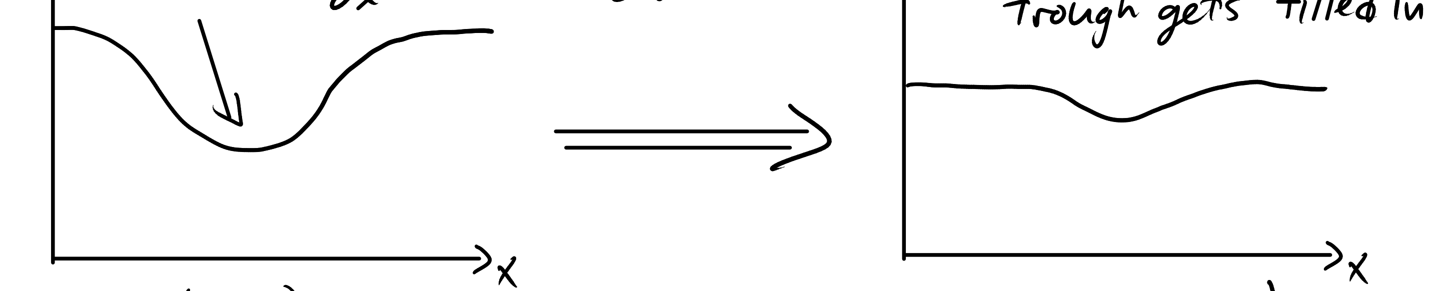
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

- $k>0$ is a constant called the "<u>thermal diffusivity</u>". It depends on the properties of the material.

Why this equation? It can be derived from the assumption that "heat flows <u>from</u> hot <u>to</u> cold."
We will not give a complete derivation, but we can understand it qualitatively



$u(x,0)$ $\qquad$ $u(x, \text{later time})$



$u(x,0)$ $\qquad$ $u(x, \text{later time})$

More specific setup: Boundary and initial conditions.

Let's consider a rod of length L: $0<x<L$

$u(x,t)$ is defined for $0<x<L$, what happens at the ends? Some possibilities:

1. Temperature is held constant: $u(0,t) = C_0$ OR
2. End is insulated, no heat escapes: $\frac{\partial u}{\partial x}(0,t) = 0$

Similar possibilities at other end $u(L,t) = C_1$ OR $\frac{\partial u}{\partial x}(L,t) = 0$

These are called "Boundary conditions". We choose them when we set up the problem.

We also need to know the initial temperature when $t=0$. This means we need to give a function $f(x)$ defined when $0<x<L$, and we want $u(x,0)=f(x)$.
$f(x)$ is called the "initial data"

Possible "full set ups" / "Well-posed problems"

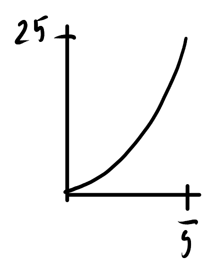
$$
* \left\{ \begin{array}{ll} \text{Domain: } 0<x<5 & \\ \frac{\partial u}{\partial t} = 10\frac{\partial^2 u}{\partial x^2} & (1) \\ u(0,t)=0 & (2) \\ u(5,t) = 25 & (3) \\ u(x,0) = x^2 & (4) \end{array} \right\}
$$

$L=5$

(1): Heat equation with $k=10$

(2)+(3): Boundary conditions with fixed temperature at endpoints

(4): initial temperature data.



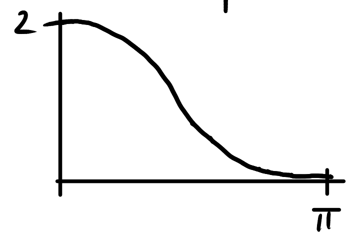
$$
* \begin{cases} \text{Domain: } 0 < x < \pi \\ \dfrac{\partial u}{\partial t} = 7 \dfrac{\partial^2 u}{\partial x^2} \quad (1) \\ \dfrac{\partial u}{\partial x}(0,t) = 0 \quad (2) \\ \dfrac{\partial u}{\partial x}(\pi,t) = 0 \quad (3) \\ u(x,0) = 1 + \cos x \quad (4) \end{cases}
$$

$L = \pi$

(1) Heat eqn with $k = 7$

(2)+(3): Boundary conditions for insulated ends.

(4): initial temperature data



In either of these situations (and in other combinations) the differential equation determines $u(x,t)$ for all $0<x<L$ and $t>0$.

We want to find this solution $u(x,t)$ analytically.

Outline of the strategy

1. Find some relatively simple solutions $u(x,t) = X(x)T(t)$
   "Separation of variables". This leads to an eigenvalue problem.

2. Combine simple solutions using principle of superposition.
   Get general solution, which is a series.

3. Use Fourier series to match initial data with the general solution.

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1. Separation of variables:

Morpheus sez: "What if I told you that $u(x,t) = g(t)\sin x$ solves $\dfrac{\partial u}{\partial t} = 5\dfrac{\partial^2 u}{\partial x^2}$?"

Then $\dfrac{\partial^2 u}{\partial x^2} = -g(t)\sin x$, $\dfrac{\partial u}{\partial t} = g'(t)\sin x$ so $g'(t) = -5g(t)$

The solution of $g'(t) = -5g(t)$ is $g(t) = Ce^{-5t}$ Constant

So $u(x,t) = Ce^{-5t}\sin x$ solves the heat equation ($k=5$)!

Another one: $e^{-20t}\sin 2x$. How far can we get with this?

* We seek solutions of $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ of the form

$$u(x,t) = X(x)\,T(t)$$

factorizable as a product of function of x and function of t.

Remark: these are very special solutions, "most" solutions are not factorizable like this.

If $u(x,t) = X(x)T(t)$ then

$$\frac{\partial u}{\partial t} = X(x)\frac{dT}{dt}(t) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{d^2X}{dx^2}T(t)$$

$\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ becomes $X(x)\dfrac{dT}{dt}(t) = k\dfrac{d^2X}{dx^2}(x)T(t)$

Divide by $kX(x)T(t)$: $\dfrac{1}{kT(t)}\dfrac{dT}{dt}(t) = \dfrac{1}{X(x)}\dfrac{d^2X}{dx^2}(x)$

This equation must be true for every value of x and t.
But left-hand side doesn't depend on x!
But right-hand side doesn't depend on t!
So neither side depends on either x or t!!!
So both sides are equal to a constant
We call it $-\lambda$, the "separation constant"

$$\frac{1}{kT}\frac{dT}{dt} = -\lambda = \frac{1}{X}\frac{d^2X}{dx^2}$$

$\Longrightarrow$ get 2 ordinary differential eqns!