

Eigenvalues and Eigenfunctions

Last time: End point value problem

$$\left\{ \begin{array}{l} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{array} \right\} \quad \begin{array}{l} \text{If } \lambda > 0 \text{ there are two possibilities} \\ \cdot \lambda = (n\pi)^2 \text{ where } n \text{ is an integer} \\ \text{Then we have} \\ y(x) = C \sin n\pi x \text{ as a solution} \\ \text{for any constant } C \end{array}$$

• $\lambda \neq (n\pi)^2$, the only solution is $y(x) \equiv 0$.
(y is constant equal to 0)

By convention, we declare the function $y(x) \equiv 0$ which is constant by equal to zero to be "uninteresting" or "trivial" or "nugatory".

All other functions are called "interesting" or "nontrivial".

So our problem only has interesting/nontrivial solutions if $\lambda = (n\pi)^2$ for some integer n .

The values of λ for which the problem has interesting/nontrivial solutions are called eigenvalues.

The interesting/nontrivial solutions themselves are called eigenfunctions or modes.

The set of eigenvalues is called the spectrum of the problem.

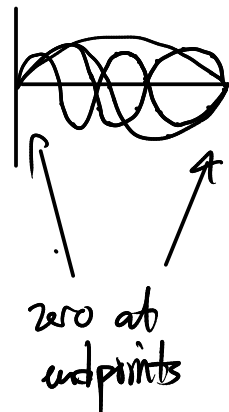
To summarize

$$\left\{ \begin{array}{l} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{array} \right\} \begin{array}{l} \text{The positive eigenvalues are} \\ \lambda = \pi^2, (2\pi)^2, (3\pi)^2, \dots \\ \text{That is } \lambda = (n\pi)^2 \quad n=1, 2, 3, \dots \end{array}$$

Associated to the eigenvalue $\lambda_n = (n\pi)^2$, we have the basic eigenfunction:

$$y_n(x) = \sin n\pi x$$

the other eigenfunctions are $Cy_n(x) = C \sin n\pi x$ where C is any constant.



What about $\lambda \leq 0$?

Is $\lambda = 0$ an eigenvalue?

$$\left\{ \begin{array}{l} y'' = 0 \\ y(0) = 0 \\ y(1) = 0 \end{array} \right\}$$

$$y'' = 0 \Rightarrow y = c_1 + c_2 x$$

$$y(0) = 0 \Rightarrow 0 = c_1 \Rightarrow y = c_2 x$$

$$y(1) = 0 \Rightarrow 0 = c_2 \Rightarrow y \equiv 0 \Rightarrow y \text{ is trivial}$$

So 0 is not an eigenvalue.

What about $\lambda < 0$?

$$\begin{array}{l} y'' + \lambda y = 0 \\ r^2 + \lambda = 0 \end{array}$$

$$r = \pm \sqrt{-\lambda}$$

this is real

$$y = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

$$y(0) = 0 \Rightarrow c_1 + c_2 = 0$$

$$y(1) = 0 \Rightarrow c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}} = 0 \Rightarrow c_1 e^{2\sqrt{\lambda}} + c_2 = 0$$

$$\Rightarrow c_1 e^{2\sqrt{\lambda}} - c_1 = 0 \Rightarrow c_1 (e^{2\sqrt{\lambda}} - 1) = 0 \Rightarrow c_1 = 0$$

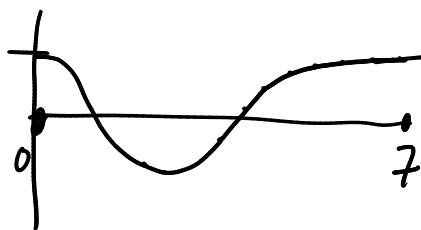
$$\Rightarrow c_2 = -c_1 = 0 \text{ so } y \equiv 0 \Rightarrow y \text{ is trivial}$$

So $\lambda < 0$ is not an eigenvalue.

We can also study several variations on this problem. We'll keep the differential equation the same, but change the endpoint conditions.

Vanishing derivative at endpoints

$$\begin{cases} y'' + \lambda y = 0 & (1) \\ y'(0) = 0 & (2) \\ y'(7) = 0 & (3) \end{cases}$$



curve should have horizontal tangent at endpoints.

What are eigenvalues and eigenfunctions?
Consider $\lambda > 0$.

$$(1) y'' + \lambda y = 0 \Rightarrow y = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$y' = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} x + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} x$$

$$(2) y'(0) = 0 \Rightarrow -c_1 \sqrt{\lambda} \cdot 0 + c_2 \sqrt{\lambda} \cdot 1 = 0 \\ \Rightarrow c_2 = 0$$

$$\text{So } y = c_1 \cos \sqrt{\lambda} x$$

$$\text{What about (3)? } y'(7) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} \cdot 7) = 0$$

This will force $c_1 = 0$ unless $\sin(\sqrt{\lambda} \cdot 7) = 0$

$$\sin(\sqrt{\lambda} \cdot 7) = 0 \Leftrightarrow \sqrt{\lambda} \cdot 7 = n\pi \text{ for some integer } n \\ \Leftrightarrow \lambda = \left(\frac{n\pi}{7}\right)^2$$

Thus if $\lambda = \left(\frac{n\pi}{7}\right)^2$ we have a nontrivial solution
 $y = C_1 \cos\left(\frac{n\pi}{7}x\right) \Rightarrow \lambda$ is an eigenvalue

Otherwise, the constant C_1 must be zero, and then $y=0$
 is the only solution. $\Rightarrow \lambda$ is not an eigenvalue

What about $\lambda=0$? $y''=0 \Rightarrow y = C_1 + C_2x \Rightarrow y' = C_2$
 $y'(0)=0 \Rightarrow C_2=0 \Rightarrow y=C_1 \Rightarrow y'=0$
 so $y'(7)=0$ automatically!

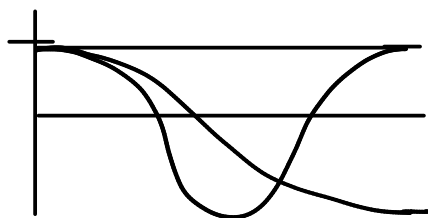
$\lambda=0$ is an eigenvalue, the eigenfunction is $y=C$ constant

Summary $\left\{ \begin{array}{l} y'' + \lambda y = 0 \\ y'(0) = 0 \\ y'(7) = 0 \end{array} \right\}$ The eigenvalues are
 $\lambda = 0, \left(\frac{\pi}{7}\right)^2, \left(\frac{2\pi}{7}\right)^2, \left(\frac{3\pi}{7}\right)^2, \dots$
 $\lambda_n = \left(\frac{n\pi}{7}\right)^2 \quad n = 0, 1, 2, 3, \dots$

Associated to the eigenvalue $\lambda_n = \left(\frac{n\pi}{7}\right)^2$ is the basic eigenfunction

$$y_n(x) = \cos \frac{n\pi x}{7} \quad (y_0(x) = \cos 0 = 1)$$

The other eigenfunctions are $C y_n(x) = C \cos \frac{n\pi x}{7}$ for C constant.



horizontal tangents at both ends.

Other variations

$$\left\{ \begin{array}{l} y'' + \lambda y = 0 \\ y(0) = 0 \\ y'(1) = 0 \end{array} \right\}$$

