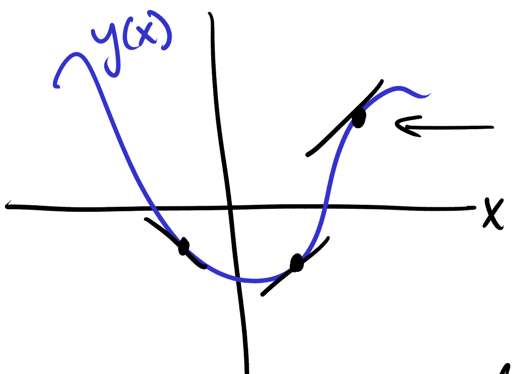
Slope fields and Solution curves
(Geometric approach to DEs)

lets consider a first-order Differential equation

$$\frac{dy}{dx} = f(x,y) \quad \text{e.g.} \quad \frac{dy}{dx} = y^2 , \quad \frac{dy}{dx} = xy$$

Can't directly indegrate because right hand side depends on y, the unknown function.

What does the equation mean geometrically

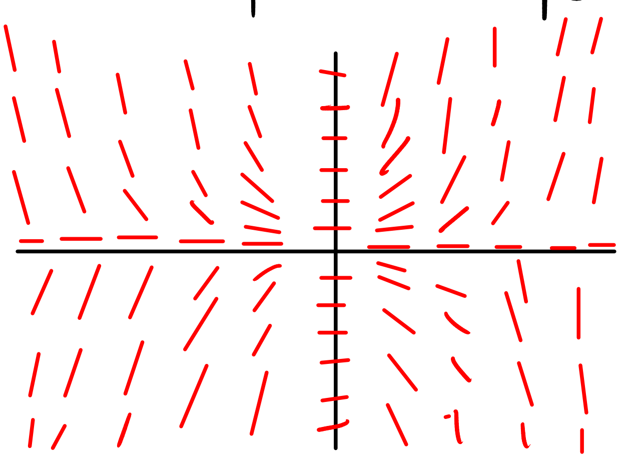


slope $= f(x,y(x))$

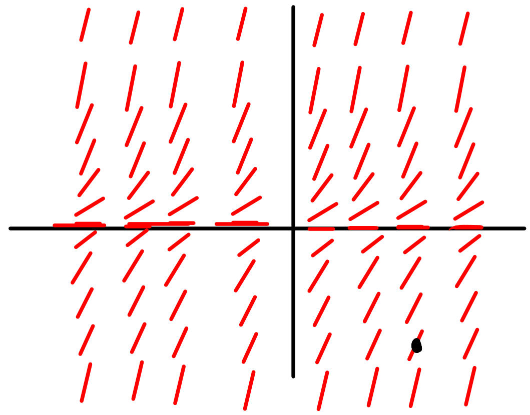
That is, the solution curve $y(x)$ (whatever it may be) must have at each point a slope $f(x,y(x))$ depending on x and the value of the solution $y(x)$, that is, depending on where we are in the xy-plane

<u>Idea</u>: Plot all the possible slopes $\Longrightarrow$ get <u>Slope field</u>

$f(x,y)=xy$



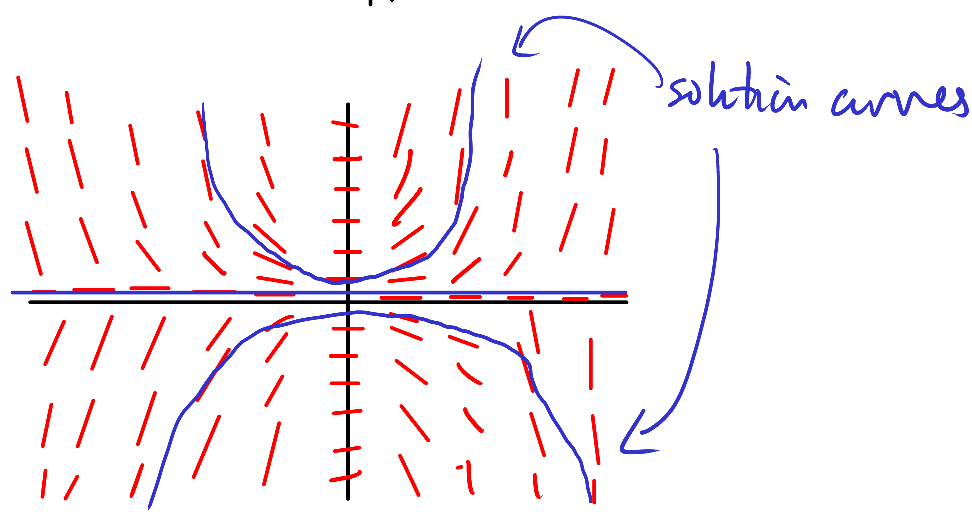
$f(x,y) = y^2$



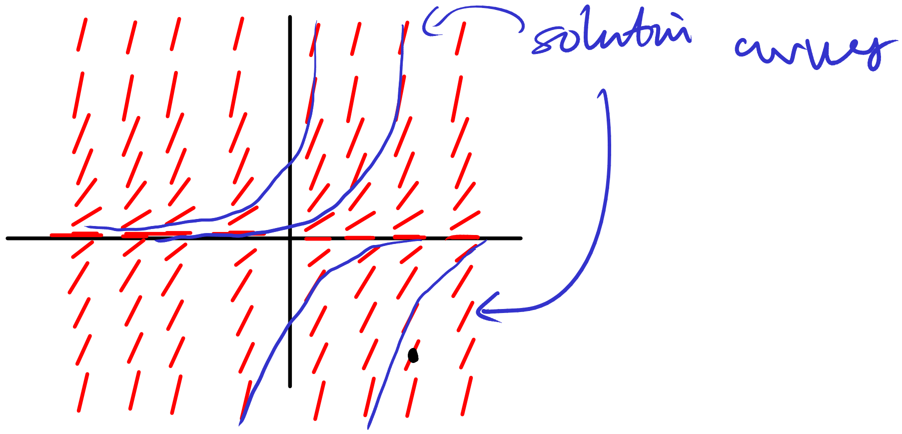
The marking means that <u>if</u> the solution passes through a particular point <u>then it must</u> have the prescribed slope. (but we don't know at the start where the solution goes.

We can use slope field to approximately sketch solutions

$f(x,y) = xy$

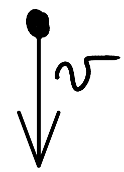


$f(x,y) = y^2$

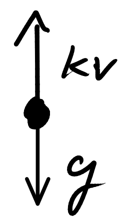


Slope fields are useful to get Qualitative info about solutions:

<u>Example</u> Object falling with air resistance let's take downward velocity to be positive. The simplest model is when air resistance is proportional to velocity

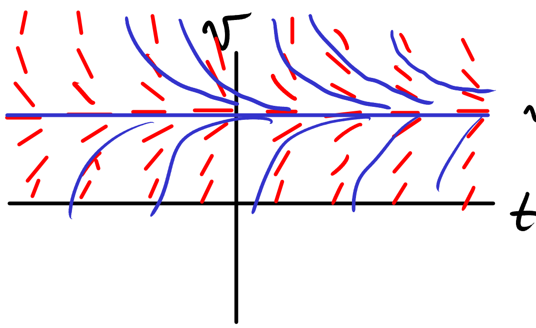


$$a = \frac{dv}{dt} = g - kv$$



Lets say $g = 9.8\ m/s^2$ and $k = .1\ s^{-1}$

The equation is $\frac{dv}{dt} = f(t,v)$ so plot slope field in (t,v)



$v=98$ $f = 0$ if $v = g/k = 98$

All solution curves asymptote to the constant solution $v = 98\ m/s$ ← called terminal velocity.

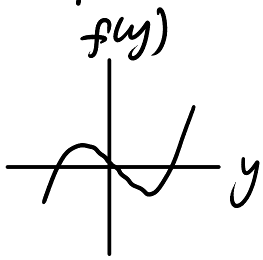
More generally, we can consider

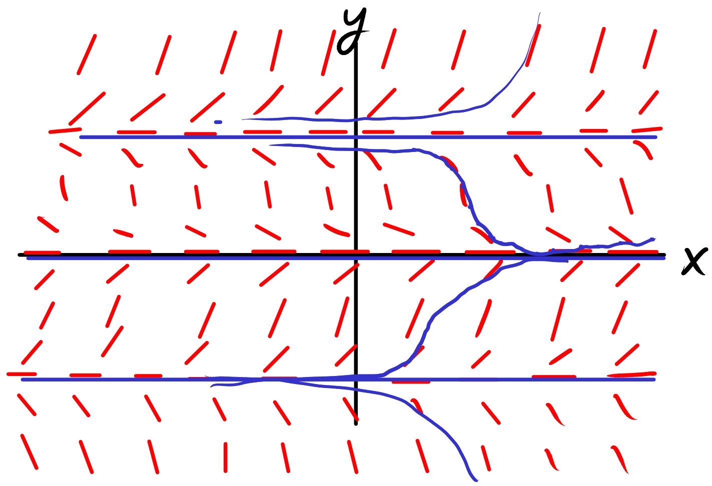
AUTONOMOUS FIRST ORDER Diff. Eq. $\frac{dy}{dx} = f(y)$ (no x on RHS)

Then the slope field only varies in the vertical direction

Example $\frac{dy}{dx} = y^3 - y = y(y+1)(y-1)$

Slope field





y values for which $f(y) = 0$ are called <u>equilibrium points</u>.

Each equilibrium point corresponds to a <u>constant solution</u>.

An equilibrium is called <u>stable</u> if nearby solutions converge to it.

In example, $y = 0$ is stable equilibrium
$y = 1$ and $y = -1$ are unstable.

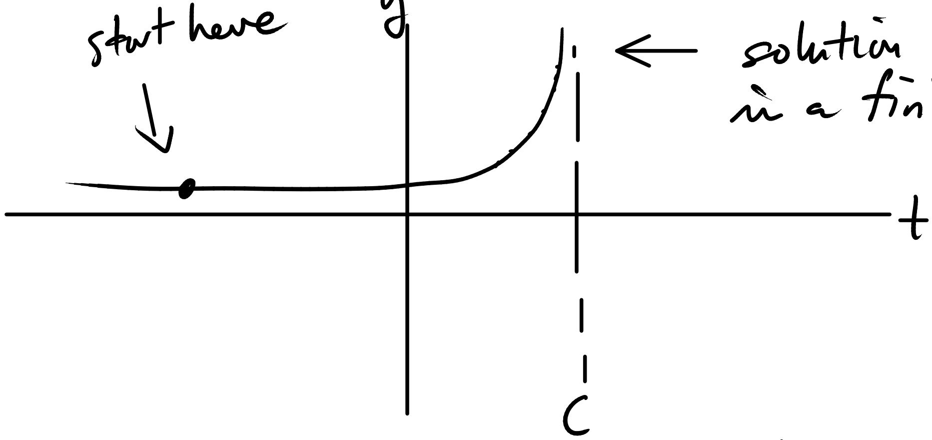
Existence and uniqueness of solutions.
In most of this course, we won't have to worry about this, but for general differential equations things can go wrong:

1. <u>Failure of existence</u> $\frac{dy}{dx} = \frac{1}{x} \implies y = \ln|x| + C$

No solution curve satisfying $y(0) = a$, or even with $y(0)$ defined!

2. <u>Eventual failure of existence</u> $\frac{dy}{dt} = y^2$

Solutions are $y(t) = \frac{1}{C-t}$ or $y(t) = 0$



← solution blows up (goes to ∞) in a finite amount of time

The solution cannot be extended to a continuous function for $t \geq C$.

3. <u>Failure of uniqueness</u> $\frac{dy}{dx} = \frac{2y}{x} \Rightarrow y(x) = Cx^2$

All values of C will satisfy initial condition $y(0)=0$!

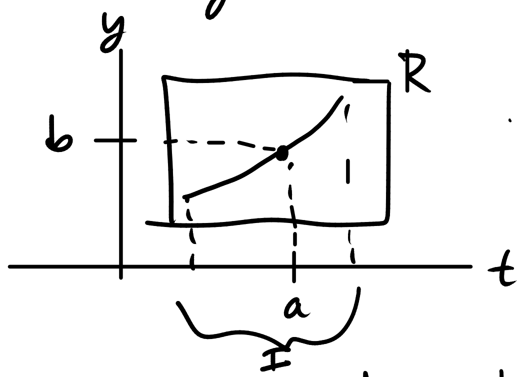
There are some natural conditions that rule out 1. and 3. (but Not 2.!)

Consider $\frac{dy}{dt} = f(t,y)$.

Informally: if $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are <u>continuous</u>

then a solution of $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(a) = b \end{cases}$ exists for times close to $t=a$, and the solution is unique.

More formally Theorem of Existence and Uniqueness



Initial value problem

$$\left\{ \begin{array}{l} \dfrac{dy}{dt} = f(t,y) \\ y(a) = b \end{array} \right\}$$

Suppose $f(t,y)$ and $\dfrac{\partial f}{\partial y}(t,y)$ are continuous on a rectangle containing (a,b).

Then there is an interval I containing a such that the initial value problem has exactly one solution on I.

Note that this theorem doesn't say anything about long-time existence. (Example 2.)