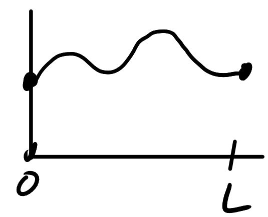
Fourier Sine and Cosine series; applications

[Review even and odd functions, see last page of previous lecture]

We will sometimes apply Fourier series methods with functions that are <u>not</u> period, but rather, defined on an interval. The idea is to extend to a period (even or odd) function, and then take the Fourier series expansion.

Let $f(t)$ be defined on the interval $[0, L]$
It's not periodic or anything like that.

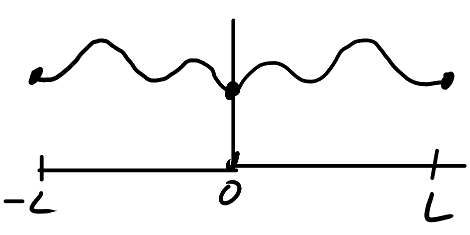


To get a Fourier series, we <u>extend</u> $f(t)$ to a periodic function on the whole line.

There are several choices of how to do this.

1. Even extension of period $2L$

Define $f_{even}(t)$ for $-L<t<0$ by $\underset{\text{in } [-L,0]}{f_{even}(t)} = \underset{\text{in } [0,L]}{f(-t)}$



Then make it periodic with period $2L$.
Since the extended function is even, it's Fourier series is a cosine series

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt \qquad a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt$$

$$f_{Even}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} \qquad (\text{Domain } -\infty<t<\infty)$$

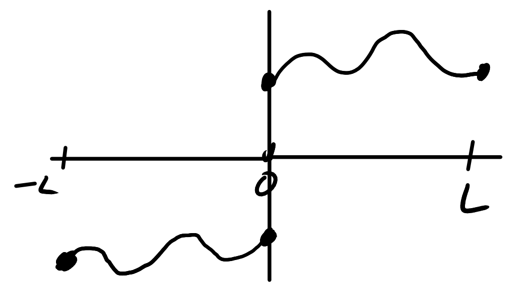
Then we can forget about values of t outside of $[0, L]$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} \quad (\text{Domain } 0 \le t \le L)$$

Thus we have represented the original function as a cosine series

2. Odd extension of period $2L$

Define $f_{odd}(t)$ for $-L < t < 0$ by $f_{odd}(t) = -f(-t)$



Then repeat with period $2L$

Since extended function is odd, Fourier series is a sine series

$$b_n = \frac{2}{L} \int_0^L f(t) \sin \frac{n\pi t}{L} \, dt$$

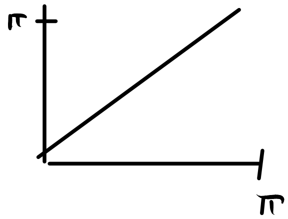
$$f_{odd}(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L} \quad (\text{Domain } -\infty < t < \infty)$$

Considering only t in $[0, L]$:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L} \quad (\text{Domain } 0 \le t \le L)$$

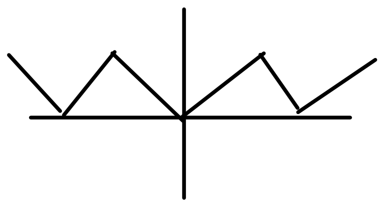
The same function $f(t)$ on $[0, L]$ has both a cosine series and a sine series.

Example: $f(t)$



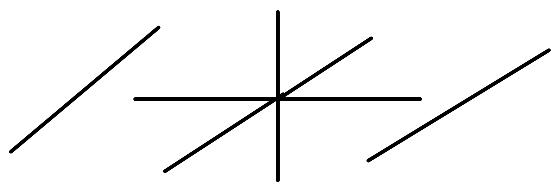
$f(t) = t$ on $0 < t < \pi$

$f_{even}(t)$ triangle!



$$f(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} \cos nt$$
$(0 < t < \pi)$

$f_{odd}(t)$ sawtooth!



$$f(t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt$$
$(0 < t < \pi)$

So the equations

$$t = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

and

$$t = 2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \cdots\right)$$

are both valid for $0 < t < \pi$!

Application of Fourier series: Solving Forced oscillator

$mx'' + kx = F(t)$ for any periodic driving force $F(t)$.

Suppose $F(t)$ is periodic with period $2L$
Expand $F(t)$ as a Fourier series

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L}\right)$$

We expect a "steady periodic" solution, that also has a Fourier series

$$x_{sp}(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi t}{L} + B_n \sin \frac{n\pi t}{L} \right)$$

Now the Fourier coefficients of $x_{sp}(t)$ play the role of "undetermined coefficients".

Plug this undetermined series into $mx'' + kx$

$$\frac{kA_0}{2} + \sum_{n=1}^{\infty} \left[k - m\left(\frac{n\pi}{L}\right)^2\right] A_n \cos \frac{n\pi t}{L} + \left[k - m\left(\frac{n\pi}{L}\right)^2\right] B_n \sin \frac{n\pi t}{L}$$

Want this equals $F(t)$, so we need

$$\frac{kA_0}{2} = \frac{a_0}{2} \qquad A_0 = \frac{a_0}{k}$$

$$\left[k - m\left(\frac{n\pi}{L}\right)^2\right] A_n = a_n \qquad A_n = \frac{a_n}{\left[k - m\left(\frac{n\pi}{L}\right)^2\right]}$$

$$\left[k - m\left(\frac{n\pi}{L}\right)^2\right] B_n = b_n \qquad B_n = \frac{b_n}{\left[k - m\left(\frac{n\pi}{L}\right)^2\right]}$$

This works as long as none of the denominators $k - m\left(\frac{n\pi}{L}\right)^2$ equals zero. Otherwise, <u>resonance</u> occurs.