

Starting Fourier series

Today: Orthogonality and Fourier coefficients

Motivation: Fourier series are useful for any situation where there is periodic behavior.

We try to write $f(t)$ as a series (= infinite sum) of sines and cosines:

$$f(t) = C + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right)$$

Then we can operate on $f(t)$ term-by-term.

Some places where this is useful:

Forced oscillation:

$$m \frac{d^2 x}{dt^2} + kx = f(t)$$

Can get a solution by adding solutions for

$$m \frac{d^2 x}{dt^2} + kx = a_n \cos \frac{n\pi t}{L} \quad m \frac{d^2 x}{dt^2} + kx = b_n \sin \frac{n\pi t}{L}$$

Solving partial differential equations of physics.

Heat equation: $u(x,t)$: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ $k = \text{thermal diffusion constant}$

Wave equation $u(x,t)$: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $c = \text{speed of wave}$

Schrödinger equation $\Psi(x,t)$ (free particle) : $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2}$ $\hbar = \text{Planck's const.}$

The orthogonality of sines and cosines at different frequencies.

Consider the sine and cosine functions

$\sin t$	angular freq = 1	$\sin 3t$	etc
$\cos t$	freq = $\frac{1}{2\pi}$	$\cos 3t$	
	Period = 2π		
$\sin 2t$	angular freq = 2	$\sin nt$	angular freq = n
$\cos 2t$	freq = $\frac{1}{\pi}$	$\cos nt$	freq = $\frac{n}{2\pi}$
	Period = π		Period = $\frac{2\pi}{n}$

Q: Take the product of two such functions, and integrate over one fundamental period of length 2π

Consider two integers m, n

$$\int_{-\pi}^{\pi} \cos mt \cos nt dt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \sin nt dt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \cos nt dt = 0 \quad \text{for all } m \text{ and } n$$

We say that $\sin nt$ and $\cos nt$ form a family of orthogonal trigonometric functions with fundamental period 2π

These identities are not extremely difficult to prove, but let's set that aside and understand what is going on.

The set of functions on an interval $[a, b]$ has a version of dot product, called inner product

Let $f(t)$ and $g(t)$ be two functions defined on $[a, b]$
Their inner product is

$$f \cdot g = \langle f, g \rangle = \int_a^b f(t)g(t) dt$$

$$\left\{ \begin{array}{l} \text{Compare: } \vec{u} = (u_1, \dots, u_n) \\ \vec{v} = (v_1, \dots, v_n) \end{array} \quad \vec{u} \cdot \vec{v} = \langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^n u_i v_i \right\}$$

Two functions are called orthogonal on $[a, b]$ if $f \cdot g = 0$

$$\text{That is, } f, g \text{ orthogonal} \iff \int_a^b f(t)g(t) dt = 0$$

So we are saying

$$\langle \cos mt, \cos nt \rangle = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\langle \sin mt, \sin nt \rangle = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\langle \cos mt, \sin nt \rangle = 0 \text{ always.}$$

Where the fundamental period is 2π .

Can use this to compute some seemingly difficult integrals

$$\int_{-\pi}^{\pi} (\sin t + \cos 3t)(\cos 4t + \sin 6t) dt$$

It's zero!

$$\int_{-\pi}^{\pi} (\underbrace{\sin t \cos 4t}_{\text{orthogonal}} + \underbrace{\sin t \sin 6t}_{\text{orthogonal}} + \underbrace{\cos 3t \cos 4t}_{\text{orthogonal}} + \underbrace{\cos 3t \sin 6t}_{\text{orthogonal}}) dt$$

integrating over one fundamental period \Rightarrow get zero!

Similar story for any fundamental period $= 2L$

Orthogonal functions: $\cos \frac{n\pi t}{L}$, $\sin \frac{n\pi t}{L}$, $n = 1, 2, 3, \dots$

\uparrow \uparrow
 period $= \frac{2\pi}{\frac{n\pi}{L}} = \frac{2L}{n}$ divides $2L$ evenly

$$\int_{-L}^L \cos \frac{n\pi t}{L} \cos \frac{m\pi t}{L} dt = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^L \sin \frac{n\pi t}{L} \sin \frac{m\pi t}{L} dt = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^L \sin \frac{n\pi t}{L} \cos \frac{m\pi t}{L} dt = 0 \text{ always.}$$