Forced oscillations II

Damped case: $mx'' + cx' + kx = F_0 \cos \omega t$

Undetermined coefficients suggests to try

$$x(t) = A\cos\omega t + B\sin\omega t$$

$$x'(t) = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$x''(t) = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$$

$$mx'' + cx' + kx = \left[(k-m\omega^2)A + c\omega B\right]\cos\omega t + \left[(k-m\omega^2)B - c\omega A\right]\sin\omega t$$

So

$$\begin{aligned} (k-m\omega^2)A + c\omega B &= F_0 \\ -c\omega A + (k-m\omega^2)B &= 0 \end{aligned}$$

$$B = \frac{c\omega}{k-m\omega^2}A$$

$$(k-m\omega^2)A + \frac{(c\omega)^2}{(k-m\omega^2)}A = F_0$$

$$\left[(k-m\omega^2)^2 + (c\omega)^2\right]A = (k-m\omega^2)F_0$$

$$A = \frac{(k-m\omega^2)F_0}{(k-m\omega^2)^2 + (c\omega)^2} \Rightarrow B = \frac{c\omega F_0}{(k-m\omega^2) + (c\omega)^2}$$

We can write $A\cos\omega t + B\sin\omega t$

as $C\cos(\omega t - \alpha)$

where $C = \sqrt{A^2 + B^2}$ $\qquad \tan\alpha = \frac{B}{A}$

Most important is C = amplitude

$$C^2 = A^2 + B^2 = \frac{(k - m\omega^2)^2 F_0^2 + (c\omega)^2 F_0^2}{\left((k-m\omega^2)^2 + (c\omega)^2\right)^2}$$

$$= \frac{F_0^2}{(k-m\omega^2)^2 + (c\omega)^2}$$

$$C = \text{Amplitude} = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$$

When the damping c is not zero, the denominator is <u>always</u> positive.

$$c \neq 0 \Rightarrow (c\omega)^2 > 0 \Rightarrow (k-m\omega^2)^2 + (c\omega)^2 > 0$$

So strictly speaking, there is no "resonance" in the damped system.

But we can still plot C vs. ω

if ω very small, $C \approx \frac{F_0}{k}$

if ω very large $C \approx 0$

When does $C = \dfrac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}$ have a maximum?

Need $\sqrt{(k-m\omega^2)^2+(c\omega)^2}$ has a minimum

Need $(k-m\omega^2)^2+(c\omega)^2$ has a minimum

$$0 = \frac{d}{d\omega}\left[(k-m\omega^2)^2+(c\omega)^2\right] = 2(k-m\omega^2)(-2m\omega)+2c^2\omega$$

$\omega=0$ is a solution, we want another one, so cancel ω

$$0 = -4m(k-m\omega^2)+2c^2$$

$$0 = -2m(k-m\omega^2)+c^2$$

$$(k-m\omega^2) = \frac{c^2}{2m}$$

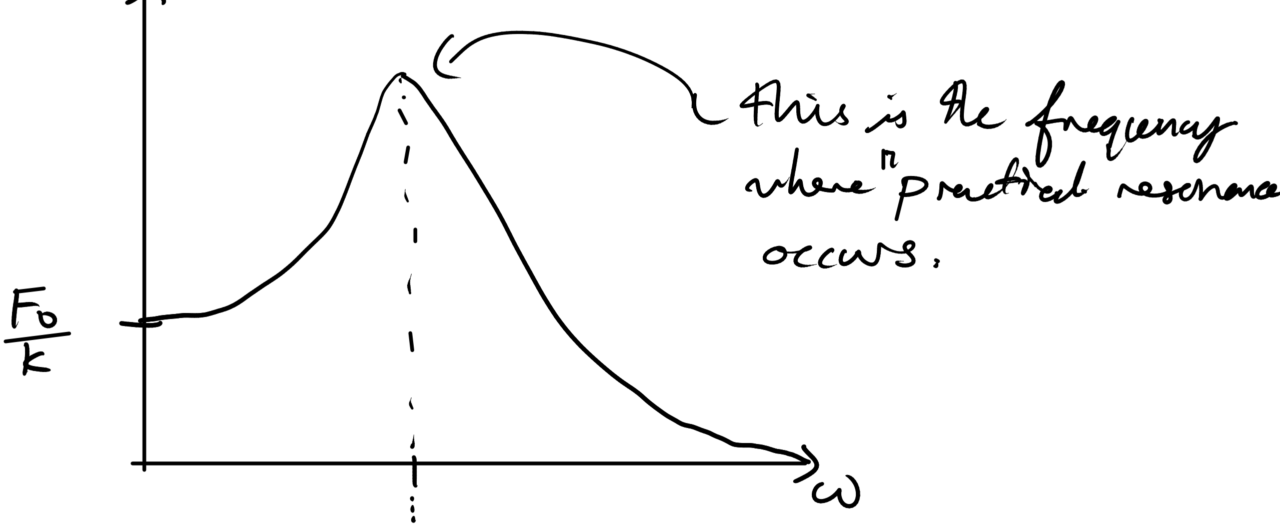
$$k - \frac{c^2}{2m} = m\omega^2$$

$$\frac{k}{m} - \frac{c^2}{2m^2} = \omega^2$$

$$\boxed{\begin{aligned}\omega_0 &= \sqrt{\frac{k}{m}}\\ \gamma &= \frac{c}{2m}\end{aligned}}$$

$$\omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} = \sqrt{\omega_0^2 - 2\gamma^2}$$

This frequency may not be real, but if it is, the plot looks like



Beats: let's go back to the undamped case.

$mx'' + kx = F_0 \cos\omega t$ Assume no resonance $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$

Particular solution $= P \cos\omega t$ where $P = \frac{F_0}{(k - m\omega^2)}$

General solution of homogeneous equation

$$= C \cos(\omega_0 t - \alpha)$$

So the general solution of the forced system is

$$x(t) = C\cos(\omega_0 t - \alpha) + P \cos\omega t$$

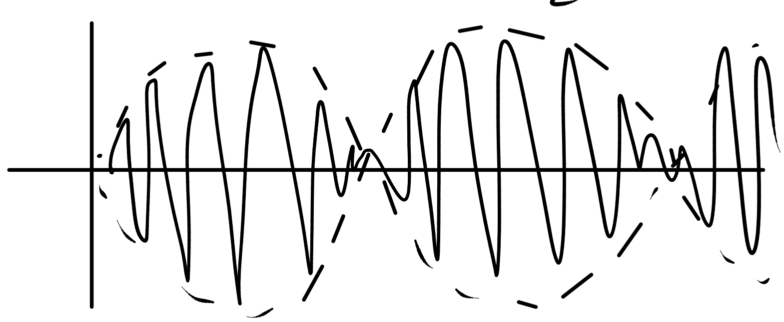
Now we use a rare trig identity "Sum-to-product"

$$\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$$

For simplicity consider $C(\cos\omega_0 t + \cos\omega t)$

$$\cos\omega_0 t + \cos\omega t = 2\cos\left(\frac{\omega_0+\omega}{2}t\right)\cos\left(\frac{\omega_0-\omega}{2}t\right)$$

If $\omega \approx \omega_0$ then $\frac{\omega_0+\omega}{2} \approx \omega_0$ $\frac{\omega_0-\omega}{2}$ is very small.



Beats