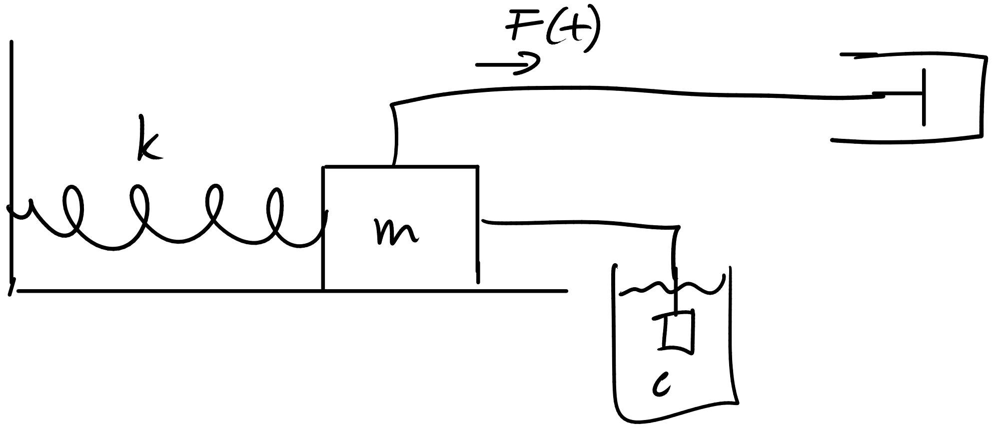
Forced oscillations I

We return to the damped oscillator, but this time we install an external driving force.



Thus Newton's 2nd law gives $ma = -kx - cv + F(t)$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

Since the system is subject to an external force, we get a <u>Nonhomogeneous equation</u>.

We will consider a periodic driving force of the form

$$F(t) = F_0 \cos(\omega t)$$
$$\text{or} \quad F(t) = F_0 \sin(\omega t)$$

ω = angular frequency (radians / second)

$\frac{\omega}{2\pi}$ = ordinary frequency (cycles / second = Hertz = Hz)

$T = \frac{2\pi}{\omega}$ = period (seconds / cycle)

remember 1 cycle $= 2\pi$ radians.

Undamped case, $c=0$:

$$m x'' + kx = F_0 \cos \omega t$$

Try undetermined coefficients

Nonhomog. term $F_0 \cos \omega t$

1st deriv $-\omega F_0 \sin \omega t$

2nd deriv $-\omega^2 F_0 \cos \omega t$

just get $\sin \omega t$ & $\cos \omega t$

So try $x(t) = A \sin \omega t + B \cos \omega t$

$$\begin{aligned} mx'' + kx &= -m\omega^2 A \sin \omega t - m\omega^2 B \cos \omega t + kA \sin \omega t + kB \cos \omega t \\ &= (-m\omega^2 + k) A \sin \omega t + (-m\omega^2 + k) B \cos \omega t \end{aligned}$$

Want this $= F_0 \cos \omega t$

Thus, $A = 0$ and $(-m\omega^2 + k) B = F_0$

so $B = \dfrac{F_0}{k - m\omega^2}$ $\qquad x_p(t) = \dfrac{F_0}{k - m\omega^2} \cos \omega t$

Recall notation $\omega_0 = \sqrt{\frac{k}{m}}$ natural frequency.

$$\frac{F_0 \;/m}{k - m\omega^2 \;/m} = \frac{F/m}{\frac{k}{m} - \omega^2} = \frac{F/m}{\omega_0^2 - \omega^2}$$

So the amplitude of the particular solution depends on the difference between the driving frequency and the natural frequency.

What happens if $\omega = \omega_0$? Driving freq = natural freq.
$k = m\omega^2$

Then we would be dividing by zero! It doesn't make sense! This is (physical) <u>resonance</u>.

And look if $k = m\omega^2$, $(mD^2 + k)(F_0 \cos \omega t)$
$= (-m\omega^2 + k) F_0 \cos \omega t = 0$

So driving force solves the homogeneous equation that is mathematical resonance that we talked about last time.

<u>Resonant case:</u> To solve $mx'' + kx = F_0 \cos(\omega_0 t)$ where $\omega_0 = \sqrt{\frac{k}{m}}$, we try

$$x(t) = At \sin \omega_0 t + Bt \cos \omega_0 t$$
$$x'(t) = A(\omega_0 t \cos \omega_0 t + \sin \omega_0 t) + B(-\omega_0 t \sin \omega_0 t + \cos \omega_0 t)$$

$$x''(t) = A(-\omega_0^2 t \sin \omega_0 t + \omega_0 \cos \omega_0 t + \omega_0 \cos \omega_0 t)$$
$$+ B(-\omega_0^2 t \cos \omega_0 t - \omega_0 \sin \omega_0 t - \omega_0 \sin \omega_0 t)$$

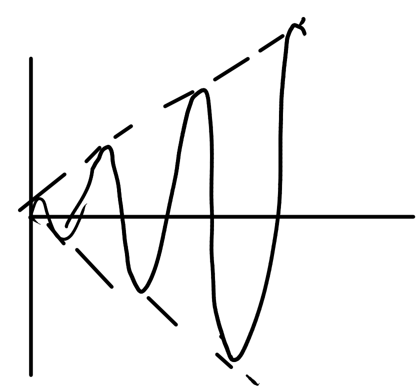
$$m x''(t) + k x(t) = m(x'' + \omega_0^2 x)$$

$$= m(A \cdot 2\omega_0 \cos \omega_0 t - B \cdot 2\omega_0 \sin \omega_0 t)$$

Want $= F_0 \cos \omega_0 t$

so $B = 0$ and $2Am\omega_0 = F_0$

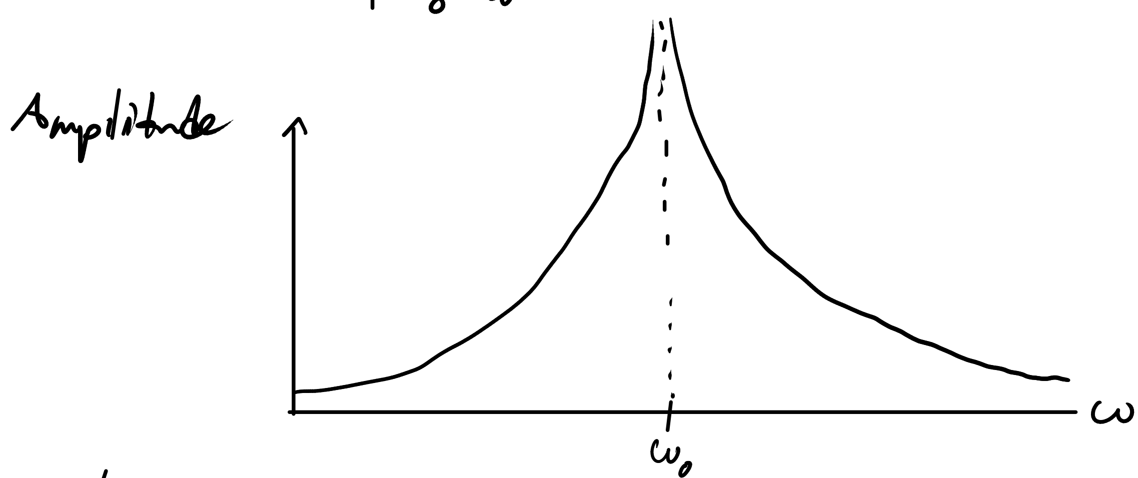
$$x_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$



We can draw a picture by plotting the amplitude of the particular solution against the frequency of the driving force

$$x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t \quad (\text{provided } \omega \neq \omega_0)$$

$$\text{Amplitude} = \left| \frac{F_0/m}{\omega_0^2 - \omega^2} \right|$$



The plot has a vertical asymptote at $\omega = \omega_0$.
This is the resonance phenomenon.

At $\omega = \omega_0$, the solution is $x_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$

It has no finite amplitude because the t factor goes to infinity!

Now, the <u>general solution</u> to the nonhomogeneous equation has other terms.

$$m x'' + kx = 0 \rightsquigarrow x(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t$$

General solution to $mx'' + kx = F_0 \cos \omega t$ $(\omega \neq \omega_0)$ is

$$x(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

or $$x(t) = C \cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

Thus the solution will be a combination of sinusoids at <u>different frequencies</u>. This leads to the phenomenon of <u>beats</u>.