Complex Roots of the Characteristic equation

Fundamental example: what is the general solution of the

Differential equation $\quad y'' + y = 0$

that is, $\quad (D^2+1)y = 0 \qquad D = \frac{d}{dx}$

Characteristic equation $\quad p(r) = r^2 + 1 = 0$

$$r^2 = -1$$

$$r = \pm\sqrt{-1} = \pm i$$

$i = \sqrt{-1}$ is called the imaginary unit

E.g. $\quad 5$: real number $\qquad 2i$: imaginary number

$5+2i$: complex number.

A complex number $a+bi$ has real part a and imaginary part b

Real part $\quad Re(5+2i) = 5$

Imaginary part $\quad Im(5+2i) = 2$

Thus $\quad r^2+1 \quad$ factors

$$p(r) = r^2+1 = (r-i)(r+i)$$

And the differential operator factors

$$p(D) = D^2+1 = (D-i)(D+i)$$

So in order to solve $\quad (D^2+1)y = 0$

we can first solve $\quad (D-i)y = 0$

$$(D+i)y = 0$$

Look at $(D-i)y=0 \iff y'-iy=0 \iff y'=iy$

Since this equation has a complex coefficient (i),
The solution cannot be an "ordinary" <u>real-valued</u> function
It must be a <u>complex-valued</u> function.

Complex-valued function. $\begin{cases} x & \text{independent variable is real} \\ y=f(x) & \text{dependent variable is allowed to be complex} \end{cases}$

The solution is $y=e^{ix}$
Does this make sense? Well by chain rule

$$y' = \frac{d}{dx}\left[e^{ix}\right] = e^{ix}\frac{d}{dx}\left[ix\right] = e^{ix}\,i = i\,e^{ix} = iy$$

In fact not only does e^{ix} exist, we have a formula for it!

EULER'S FORMULA: $\quad e^{ix} = \cos(x) + i\sin(x)$

Feynman: "This is our jewel." Ch. 22 of lectures on physics

Check $\frac{d}{dx}\left[\cos(x)+i\sin(x)\right] = -\sin(x) + i\cos(x)$

$i\left[\cos(x)+i\sin(x)\right] = i\cos(x) + i^2\sin(x)$

these are equal b/c $i^2=-1$!

Now look at $(D+i)y=0 \to y'=-iy \longrightarrow y=e^{-ix}$

$$e^{-ix} = e^{i(-x)} = \cos(-x)+i\sin(-x) = \cos(x) - i\sin(x)$$

Thus both e^{ix} and e^{-ix} solve $(D^2+1)y=0$

But these are complex valued solutions, we want real-valued ones

General principle: If a complex-valued function solves a differential equation, and the equation has real coefficients, then the real and imaginary parts both solve the same differential equation.

So because $e^{ix} = \cos(x) + i\sin(x)$ solves $\underbrace{(D^2+1)}_{\text{real}}y = 0$

We know that

$$\left.\begin{aligned} \text{Real part} \quad Re(e^{ix}) &= \cos(x) \\ \text{and Imag. part} \quad Im(e^{ix}) &= \sin(x) \end{aligned}\right\} \text{ also solve } (D^2+1)y=0$$

Indeed

$$(D^2+1)[\cos(x)] = -\cos(x) + \cos(x) = 0$$
$$(D^2+1)[\sin(x)] = -\sin(x) + \sin(x) = 0$$

Also e^{-ix} solves $(D^2+1)y=0$ so

$$\left.\begin{aligned} Re(e^{-ix}) &= \cos(x) \\ \text{and } Im(e^{-ix}) &= -\sin(x) \end{aligned}\right\} \text{ also solve, but these are not really new.}$$

General irreducible quadratic has a pair of complex conjugate roots.

Consider $ar^2+br+c=0$ Assume $b^2-4ac<0$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b}{2a} \pm i\,\frac{\sqrt{4ac-b^2}}{2a} = p \pm qi$$

where $p = \frac{-b}{2a}$ and $q = \frac{\sqrt{4ac-b^2}}{2a}$

Then we can consider the differential equation

$$(aD^2+bD+c)\,y=0$$

$$\big(D-(p+qi)\big)\big(D-(p-qi)\big)\,y=0$$

$$\boxed{\begin{aligned} p&=\frac{-b}{2a}\\ q&=\frac{\sqrt{4ac-b^2}}{2a}\end{aligned}}$$

Solution of $\big(D-(p+qi)\big)\,y=0$ is $y=e^{(p+qi)x}$

What is that? $e^{(p+qi)x}=e^{px+iqx}=e^{px}e^{iqx}$

$$=e^{px}\big(\cos(qx)+i\sin(bx)\big)$$

Since this ↗ solves $(aD^2+bD+c)y=0$, so do its real and imaginary parts

$$\left.\begin{aligned}\mathrm{Re}\left(e^{(p+qi)x}\right)&=e^{px}\cos(qx)\\ \mathrm{Im}\left(e^{(p+qi)x}\right)&=e^{px}\sin(qx)\end{aligned}\right\}\ \text{these both solve } (aD^2+bD+c)\,y=0$$

The <u>general solution</u> is $\boxed{y(x)=c_1e^{px}\cos(qx)+c_2e^{px}\sin(qx)}$

<u>Example</u> find general solution of $(D^2-D+2)\,y=0$

Characteristic equation

$$r^2-r+2=0 \qquad r=\frac{1\pm\sqrt{1-4\cdot 2}}{2}=\frac{1}{2}\pm i\frac{\sqrt{7}}{2}$$

A Complex solution is $y=e^{\left(\frac{1}{2}+i\frac{\sqrt{7}}{2}\right)x}=e^{\frac{1}{2}x}e^{i\frac{\sqrt{7}}{2}x}=$

$$=e^{\frac{1}{2}x}\left(\cos\left(\frac{\sqrt{7}}{2}x\right)+i\sin\left(\frac{\sqrt{7}}{2}x\right)\right)$$

Real and Imaginary parts $y_1=e^{\frac{1}{2}x}\cos\left(\frac{\sqrt{7}}{2}x\right)$ $\quad y_2=e^{\frac{1}{2}x}\sin\left(\frac{\sqrt{7}}{2}x\right)$

<u>Real</u> General solution: $\boxed{y(x)=c_1e^{\frac{1}{2}x}\cos\left(\frac{\sqrt{7}}{2}x\right)+c_2e^{\frac{1}{2}x}\sin\left(\frac{\sqrt{7}}{2}x\right)}$