Constant coefficient linear homogeneous equations

This is an equation like

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are <u>constants</u>.

The basic idea is to try a solution of the form e^{rx}

$$y = e^{rx} \quad y' = re^{rx} \quad y'' = r^2 e^{rx}, \ldots, \quad y^{(n)} = r^n e^{rx}$$

Then $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$

becomes

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_1 r e^{rx} + a_0 e^{rx} = 0$$

$$\left(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0\right) e^{rx} = 0$$

Thus e^{rx} will be a solution of

$$\boxed{a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0}$$ Characteristic Equation.

To get Characteristic equation from the original homogeneous linear differential equation, replace $y \to 1$, $y' \to r$, $y'' \to r^2$, etc $y^{(n)} \to r^n$

To summarize: If r solves characteristic equation, then $y = e^{rx}$ solves the differential equation.

Is this all there is to it?
If the solutions of the characteristic equation are real and distinct, then yes.

Let n = order of differential equation = degree of characteristic equation.

A degree n polynomial equation such as

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

has n solutions $r_1, \ldots, r_n$ which may be complex $(a+bi)$ and which may be repeated such that the polynomial factors as

$$a_n(r-r_1)(r-r_2)(r-r_3)\cdots(r-r_n)$$

Theorem

If the characteristic equation has n solutions $r_1, r_2, \ldots, r_n$ that are all distinct and real, then the general solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

Fact: for $r_1, r_2, \ldots, r_n$ distinct, the functions $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ are linearly independent

[Follows from the relation between the discriminant and the Vandermonde determinant, if you care.]

Example $y^{(3)} - 3y'' + 2y' = 0$

Characteristic equation $r^3 - 3r^2 + 2r = 0$

$$r(r^2 - 3r + 2) = 0$$
$$r(r-1)(r-2) = 0$$

So solutions $r_1 = 0$ $r_2 = 1$ $r_3 = 2$ Real, distinct

general solution $y(x) = c_1 e^{0x} + c_2 e^{x} + c_3 e^{2x} = c_1 + c_2 e^{x} + c_3 e^{2x}$

What if a root is repeated?

Eg $y'' - 2y' + y = 0$
$r^2 - 2r + 1 = 0$
$(r-1)^2 = 0$ ← double solution $r_1 = 1$ $r_2 = 1$

So $y = e^{r_1 x} = e^{x}$ is a solution

The general solution is <u>NOT</u> $y(x) = c_1 e^{x} + c_2 e^{x}$

because e^x and e^x are <u>not</u> linearly independent!

We need ANOTHER solution that CANNOT be found using the characteristic equation.

This requires some work, so we are going to introduce some theory that will make it more comprehensible.

Constant coefficient differential operators.

What is $\frac{d}{dx}$? Well $\frac{d}{dx}[f(x)] = f'(x)$ derivative

The symbol "$\frac{d}{dx}$" represents the operation of differentiation

the operation that takes a function and produces its derivative

$\frac{d}{dx}$ is therefore called the "Derivative operator"

Similarly $\frac{d^2}{dx^2}$ is an operator called the "second derivative operator"

And so on $\frac{d^2}{dx^3}, \dots, \frac{d^n}{dx^n}$ = nth derivative operator.

Now we can define algebraic operations on operators then selves

If A is an operator, then A^2 does the operation twice

So if $D = \frac{d}{dx}$ then D^2 = take derivative twice $= \frac{d^2}{dx^2}$

and $D^3 = \frac{d^3}{dx^3}, \dots, D^n = \frac{d^n}{dx^n}$

We can also combine differential operators by addition

$$L = D^2 - 2D + 3$$ this is the operator that multiplies by 3

$$= \frac{d^2}{dx^2} - 2\frac{d}{dx} + 3$$

Then $$L[f(x)] = f''(x) - 2f'(x) + 3f(x)$$

A general constant coefficient linear differential operator looks like

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

$$L \cdot y = \left(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0\right) \cdot y$$

$$= a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

So this is a new notation for the LHS's of the differential equations we want to consider.