Linear independence and general solutions

We return to the question, when have we found the <u>complete</u> general solution to a linear differential equation?

<u>2nd order Homogeneous case</u> $y'' + p(x)y' + q(x)y = 0$

We must find 2 solutions, and the must be really different from each other. The precise term is linearly independent.

<u>Definition</u> Two functions $y_1(x)$ and $y_2(x)$ are <u>linearly independent</u> if they are NOT PROPORTIONAL

$$\left.\begin{array}{l} y_1(x) \neq C y_2(x) \\ y_2(x) \neq C y_1(x) \end{array}\right\} \text{ for any constant } C.$$

<u>Examples</u>
- $\sin(x)$, $\cos(x)$
- e^x, e^{-2x}
- e^x, xe^x

<u>Nonexamples</u>
- $e^x, 2e^x$
- $0, \sin(x)$

The complete general solution of a second order linear homogeneous differential equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1 and y_2 are two linearly independent solutions

Eg. for $y'' - 4y = 0$

$y = c_1 e^{2x} + c_2 e^{-2x}$ is a complete general solution

So is $y = d_1 \sinh(2x) + d_2 \cosh(2x)$

or even $y = k_1 e^{2x} + k_2 \sinh(2x)$

Higher order linear differential equations

Notation: $y^{(n)} = \underbrace{y^{''\cdots'}}_{n \text{ primes}} = \dfrac{d^n y}{dx^n}$ = nth derivative of y wrt x.

A typical n-th order linear DE looks like

$$y^{(n)} + p_1(x) y^{(n-1)} + p_2(x) y^{(n-2)} + \cdots + p_{n-1}(x) y' + p_n(x) y = q(x)$$

The equation is <u>homogeneous</u> if $q(x) = 0$.

Let's do a third-order example:

$$y''' - 3y'' - y' + 3y = 0$$

Behind the scenes

$(r-1)(r+1)(r-3)$

$(r^2-1)(r-3)$

$r^3 - 3r^2 - r + 3$

For constant coefficient linear homogeneous, we can try $y = e^{rx}$ again.

$y' = re^{rx}$ $\quad y'' = r^2 e^{rx}$ $\quad y''' = r^3 e^{rx}$

$$r^3 e^{rx} - 3r^2 e^{rx} - re^{rx} + 3e^{rx} = 0$$

$\boxed{r^3 - 3r^2 - r + 3 = 0}$ CHARACTERISTIC EQN.

In this case the characteristic equation factors nicely.

$$r^3 - 3r^2 - r + 3 = (r^2-1)(r-3) = (r-1)(r+1)(r-3)$$

so the roots are $r = 1, -1, 3$

Thus $y_1 = e^x$ $y_2 = e^{-x}$ and $y_3 = e^{3x}$ are solutions of the Diff Eqn. $y''' - 3y'' - y' + 3y = 0$

Linear homogeneous Diff eqn of any order satisfy <u>principle of superposition</u>

So a general solution is $y = c_1 e^x + c_2 e^{-x} + c_3 e^{3x}$ where c_1, c_2, and c_3 are constants.

Why is this the <u>complete</u> general solution?

For a third-order equation, we need to specify 3 initial conditions

$$\begin{cases} y(a) = b_0 \\ y'(a) = b_1 \\ y''(a) = b_2 \end{cases}$$

For n-th order, we need to specify n initial conditions

$$\left.\begin{array}{l} y(a) = b_0 \\ y'(a) = b_1 \\ y''(a) = b_2 \\ \vdots \\ y^{(n-1)}(a) = b_{n-1} \end{array}\right\} \text{ n conditions.}$$

So we need n constants

$\rightarrow$ we need n distinct solutions

The n solutions need to be <u>linearly independent</u>

It easier to define when a collection of n functions is <u>linearly dependent</u> (a/k/a "<u>redundant</u>")

- A collection of n functions $y_1, \dots, y_n$ is <u>linearly dependent</u> (or <u>redundant</u>) if there exist constants $c_1, \dots, c_n$, NOT ALL $= 0$, such that
$$c_1 y_1 + c_2 y_2 + \dots + c_3 y_3 = 0 \quad \text{\underline{for all} } x.$$

- n functions $y_1 \dots y_n$ are <u>linearly</u> <u>independent</u> if they are not <u>linearly dependent</u>

Eg. e^x, e^{-x} and e^{3x} are linearly independent.

suppose
$$c_1 e^x + c_2 e^{-x} + c_3 e^{3x} = 0$$

Plug in $x=0$
$$c_1 + c_2 + c_3 = 0$$

Take derivative and plug in $x=0$
$$c_1 e^x - c_2 e^{-x} + 3c_3 e^{3x} = 0$$
$$c_1 - c_2 + 3c_3 = 0$$

Take 2nd deriv and plug in 0
$$c_1 + c_2 + 9c_3 = 0$$

$8c_3 = 0 \Rightarrow c_3 = 0$

$c_1 = c_2$

$c_1 + c_2 = 0 \rightarrow c_1 = 0$, $c_2 = 0$.

Since all coefficients are forced to be zero, the functions are linearly independent.

Not independent: $e^x, e^{-x}, \sinh(x) = \dfrac{e^x - e^{-x}}{2}$