Second-Order Linear equations

General form $\quad A(x)\,y'' + B(x)\,y' + C(x)\,y = F(x)$

As usual, $y' = \frac{dy}{dx} \quad y'' = \frac{d^2y}{dx^2}$

e.g. $\quad x^2 y'' + \sin(x)\,y' + 13x^3 y = e^x$

Second-order because we have y''
Linear because y, y', y'' appear to first power only.

Note $y'' = yy'$ is <u>not</u> linear, because to things involving y are multiplied together.

<u>Homogeneous versus Non-homogeneous</u>

Nonhomogeneous is the general case: $\quad A(x)\,y'' + B(x)\,y' + C(x)\,y = F(x)$

Homogeneous is when the right-hand side is zero: $\quad A(x)\,y'' + B(x)\,y' + C(x)\,y = 0$

<u>Example</u>: $\quad y'' + 3y' + 2y = e^x \leftarrow$ Nonhomogeneous

$y'' + 3y' + 2y = 0 \leftarrow$ Homogeneous version.

["Homogeneous" = every term involves y'', y', or y.]

Let's talk about solutions:

<u>Example</u> $y'' - 4y = 0$

Here are some solutions

$y_1(x) = e^{2x}$: indeed $(e^{2x})'' = 2(e^{2x})' = 4e^{2x}$

so $(e^{2x})'' - 4(e^{2x}) = 0$

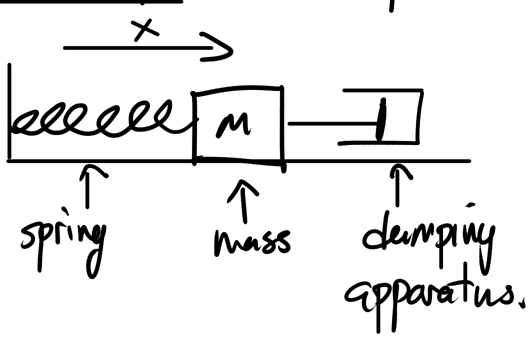
Also $y_2(x) = e^{-2x}$: indeed $(e^{-2x})'' = -2(e^{-2x})' = (-2)^2 e^{-2x}$

so $(e^{-2x})'' - 4(e^{-2x}) = 0.$

There are even other solutions, such as $\sinh(2x)$, $\cosh(2x)$, and more.

We want to understand the "space" or "set" of all possible solutions, this is our goal.

<u>Physical example</u>: Damped oscillation



Spring force $= -kx$

damping force $= -cv$

Newton's 2^{nd}: $ma = -kx - cv$

$$ma + cv + kx = 0$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

This is second order linear (and homogeneous)!

Physically, we know that we need to specify the initial position and the initial velocity.

So the general solution of a second order equation should depend on two undetermined constants, which may later be fixed by initial conditions.

FACT (to be understood)
The general solution of

$$y'' - 4y = 0$$

is

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}$$

Problem: use the "FACT" to solve the initial value problem

$$\begin{cases} y'' - 4y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Solution: Need to find c_1 and c_2

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}$$
$$y'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

$$1 = y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2$$

$$0 = 2c_1 e^0 - 2c_2 e^0 = 2c_1 - 2c_2$$

Solve for c_1 and c_2 : get $c_1 = c_2 = \frac{1}{2}$

So $y(x) = \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}$ is the particular solution to the IVP.

Principle of Superpostion
for second order linear <u>homogeneous</u> equations.

Consider $A(x)\,y'' + B(x)\,y' + C(x)\,y = 0$
Suppose two functions $y_1(x)$ and $y_2(x)$ satisfy this equation

THEN so does $c_1 y_1 + c_2 y_2$, where c_1 and c_2 are any constants.

<u>Proof</u>: We know $Ay_1'' + By_1' + Cy_1 = 0$
and $Ay_2'' + By_2' + Cy_2 = 0$

So look at
$$A(c_1y_1+c_2y_2)'' + B(c_1y_1+c_2y_2)' + C(c_1y_1+c_2y_2)$$
$$= Ac_1y_1'' + Ac_2y_2'' + Bc_1y_1' + Bc_2y_2' + Cc_1y_1 + Cc_2y_2$$
$$= c_1(Ay_1'' + By_1' + Cy_1) + c_2(Ay_2'' + By_2' + Cy_2)$$
$$= c_1 \cdot 0 + c_2 \cdot 0 = 0 \quad \text{QED}$$

<u>Example</u>: since e^{2x} and e^{-2x} solve $y'' - 4y = 0$
so do:
- $e^{2x} + e^{-2x}$
- $e^{2x} - e^{-2x}$
- $15e^{2x}$
- $\pi e^{2x} + 17e^{-2x}$ etc.