

Differential equations

- Q1: What are they?
Q2: Why would you care?
Q3: What are we doing in this course?

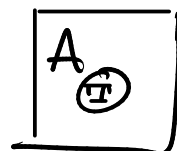
A1: A differential equation is an equation relating the derivatives of some quantities to other quantities.

A2: Since a derivative represents a rate of change, any quantitative problem involving rates of change necessarily involves differential equations of some kind.

A3: Because the concept of a differential equation is so broad, and touches so many disparate phenomena, we cannot solve all DEs. Instead we focus on specific cases that are tractable and that have interesting applications.

Example: Newton's Law of cooling

An object is immersed in a bath held at a fixed temperature A (ambient temperature)



Denote by T the temperature of the object. This is actually a function of time (t), since the object will cool or heat up to match the ambient temp A (tends toward thermodynamic equilibrium)

Newton says

$$\frac{dT}{dt} = -k(T-A)$$

k = constant measuring rate at which heat is conducted to/from the object. Depends on physical composition of the object. k is always positive

This is a differential equation because it involves the derivative $\frac{dT}{dt}$.

We want to know more explicitly what T is as a function of t . In fact

$$T(t) = C e^{-kt} + A$$

Solves the Differential equation, where C is any number.

It's easy to check: Let $T(t) = C e^{-kt} + A$

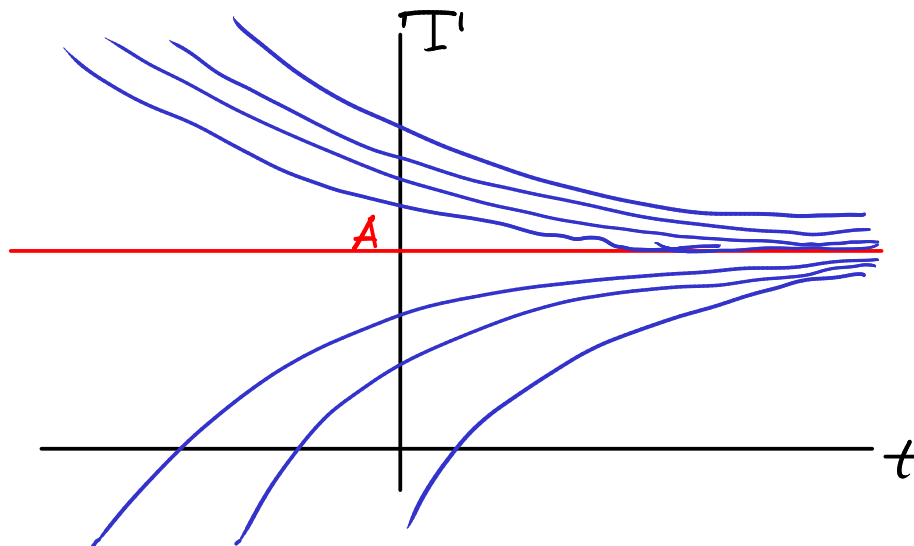
$$\text{LHS} = \frac{dT}{dt} = \frac{d}{dt}(C e^{-kt}) + \frac{d}{dt}(A) = C \frac{d}{dt}(e^{-kt}) + 0$$

$$= C(-k) e^{-kt} \quad \leftarrow \text{equal!}$$

$$\text{RHS} = -k(T-A) = -k(C e^{-kt} + A - A) = -k C e^{-kt}$$

Observe: the DE has infinitely many solutions, because there are infinitely many possibilities for the constant C . This is a typical characteristic.

Plots of possible solutions



Makes sense that $T(t)$ is not unique, because we never decided whether the object starts out above or below the Ambient temp. A .

In other words, we need to specify the initial temperature $T_0 = T(0)$.

T_0 is related to C :

$$T_0 = T(0) = C e^{-k \cdot 0} + A = C + A$$

$$\text{So } C = T_0 - A.$$

$$\text{So } T(t) = (T_0 - A) e^{-kt} + A$$

We solved the initial value problem $\frac{dT}{dt} = -k(T-A)$, $T(0) = T_0$.

To be more specific

t = time in seconds

T = temp in Kelvin

$$k = 5 \text{ s}^{-1}$$

$$A = 300 \text{ K}$$

$$T_0 = 2000 \text{ K}$$

$$\begin{aligned} T(t) &= (T_0 - A) e^{-kt} + A \\ &= 1700 e^{-5t} + 300 \end{aligned}$$

Many models involve differential equations

* Logistic model for population: $\frac{dP}{dt} = kP(C-P)$ $P(t)$ = pop.

* Harmonic oscillator:

Mass on spring



$$m \frac{d^2x}{dt^2} + kx = 0$$

$x(t)$ = position

pendulum
(approximately)



with friction:

$$m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$$

* Airy equation: $\frac{d^2y}{dx^2} = xy$ models intensity of the rainbow
explains (at least approximately)
the "double rainbow" phenomenon

What differential equation does it solve?

$$y = e^t, \quad \frac{dy}{dt} = y$$

$$y = \frac{1}{1-x}, \quad \frac{dy}{dx} = y^2$$

$$y = \sin t, \quad \frac{d^2y}{dt^2} = -y$$

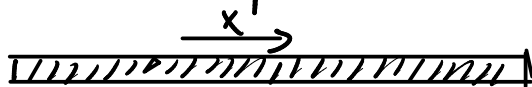
$$y = \frac{1}{(1-x)^2}, \quad \frac{dy}{dx} = 2y^{3/2}$$

What about the equation $(y')^2 + (y)^2 = -1$?
 No (real) solution!

So existence of solutions is sometimes in doubt.

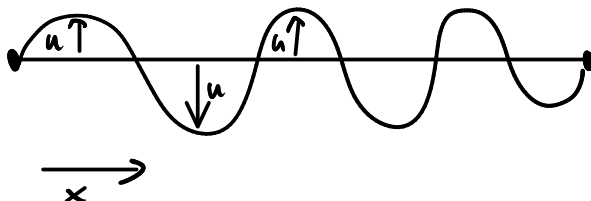
Many problems involve functions of several variables
 Then we are dealing with partial differential equations

Heat conduction: $T(x,t)$ temperature in rod



Heat Equation: $\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$

Wave propagation: $u(x,t)$ = displacement of vibrating string



Wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Fluid flow

\vec{v} = velocity vector field

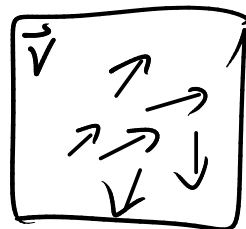
ϕ = pressure

k = viscosity \vec{f} = external force

(1) $\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = k \nabla^2 \vec{v} - \nabla \phi + \vec{f}$

(2) $\nabla \cdot \vec{v} = 0$

(3) $\vec{v}(x, 0) = \vec{v}_0(x)$ initial condition



Navier-Stokes
 \$1000000 prize
 see claymath.org